COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACE USING SEMI-COMPATIBLE MAPPINGS

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ABSTRACT

In this paper, the concept of semi-compatible and occasionally weakly compatible maps in fuzzy metric space has been introduced to prove common fixed point theorem which generalize the result of Sharma [13].

Keywords : Common Fixed Points, Fuzzy Metric Space, Compatible Maps, Occasionally Weakly Compatible Mappings, Weak Compatible Mappings And Semi-Compatible Mappings.

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I. INTRODUCTION

Zadeh [18] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. A fuzzy set A in X is a function with domain X and values in [0, 1]. Since then, many authors regarding the theory of fuzzy sets and its applications have developed a lot of literatures.

However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [10] and modified by George and Veeramani [4]. Recently, Grabiec [5] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [14] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [8] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [2, 3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh et. al. [15, 16] proved fixed point theorems in a fuzzy metric space. Recently in 2012, Jain et. al. [6, 7] and Sharma et. al. [12] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

II. PRELIMINARIES

Definition 2.1. [11] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [11] The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [11] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [11] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [14] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5. [15] Two self maps A and B of a fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, i.e. Ax =

Bx implies ABx = BAx.

Definition 2.6. Self maps A and S of a Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Definition 2.7. [6] Suppose A and S be two maps from a Fuzzy metric space (X, M, *) into itself. Then they are said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Proposition 2.1. [16] In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Proposition 2.2. [14] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and let $\{x_n\}$ be a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X. Then $STx_n \to Tu$ provided T is continuous.

Proposition 2.3. [14] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and Su = Tu for some u in X then STu = TSu = SSu = TTu.

Lemma 2.1. [5] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, M(x, y, .) is a non-decreasing function.

Lemma 2.2. [1] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)\ \forall\ t > 0$, then $x = y$.

Lemma 2.3. [16] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)\ \forall\ t > 0 \text{ and } n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4.[9] The only t-norm * satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is

$$a * b = \min\{a, b\} \text{ for all } a, b \in [0, 1].$$

III. MAIN RESULT

Now we prove the following results:

Theorem 3.1. Let (X, M, *) be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$. Let A, B, S, T, P and Q be mappings from X into itself satisfying

(3.3.1) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$;

(3.3.2) AB = BA, ST = TS, PB = BP, SQ = QS, QT = TQ;

(3.3.3) Pair (P, AB) is semi-compatible and (Q, ST) is occasionally weakly compatible;

(3.3.4) There exists a number $k \in (0, 1)$ such that

$$M(Px, Qy, kt) \geq M(ABx, Px, t) * M(STy, Qy, t) * M(STy, Px, \beta t)$$

$$* M(ABx, Qy, (2 - \beta)t) * M(ABx, STy, t),$$

for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$.

If the range of one of the subspaces P(X) or AB(X) or Q(X) or ST(X) is complete, then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. By [13], $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, so $\{y_n\}$ converges to a point $z \in X$. Since $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, they also converge to the same point z.

Since $P(X) \subset AB(X)$, there exists a point $u \in X$ such that $ABu = z$. Then, using (3.3.4)

$$M(Pu, z, kt) \geq M(Pu, Qx_{2n+1}, kt)$$

$$\geq M(ABu, Pu, t) * M(STx_{2n+1}, Qx_{2n+1}, t) * M(STx_{2n+1}, Pu, \beta t)$$

$$* M(ABu, Qx_{2n+1}, (2 - \beta)t) * M(ABu, STx_{2n+1}, t).$$

Proceeding limit as $n \to \infty$ and setting $\beta = 1$,

$$M(Pu, z, kt) \geq M(Pu, z, t) * M(z, z, t) * M(z, Pu, \beta t) * M(z, z, t) * M(z, z, t)$$

$$= M(Pu, z, t) * 1 * M(Pu, z, t) * 1 * 1;$$

$$\geq M(Pu, z, t).$$

By Lemma (2.2),

$$Pu = z.$$

Therefore, $ABu = Pu = z$.

Since $Q(X) \subset ST(X)$, there exists a point $v \in X$ such that $z = STv$. Then, again using (3.3.4)

$$M(Pu, Qv, kt) \geq M(ABu, Pu, t) * M(STv, Qv, t) * M(STv, Pu, \beta t)$$

$$* M(ABu, Qv, (2 - \beta)t) * M(ABu, STv, t)$$

Proceeding limit as $n \to \infty$, we have for $\beta = 1$, $Qv = z$.

Therefore, $ABu = Pu = STv = Qv = z$.

Since pair (P, AB) is semi-compatible, so

$$\lim_{n \to \infty} PABx_{2n} = ABz.$$

Also, $\lim_{n \to \infty} PABx_{2n} = Pz$.

Since the limit in a Fuzzy metric space is unique, we get

$$Pz = ABz.$$

Now we show that z is a fixed point of P. For $\beta = 1$, we have

$$M(Pz, Qv, kt) \geq M(ABz, Pz, t) * M(STv, Qv, t) * M(STv, Pz, \beta t)$$

$$* M(ABz, Qv, (2 - \beta)t) * M(ABz, STv, t)$$

$$= 1 * 1 * M(z, Pz, t) * M(Pz, z, t) * M(Pz, z, t).$$

Therefore, we have by Lemma 2.2,

$$Pz = z.$$

Hence

$$Pz = z = ABz:$$

Similarly, pair of map {Q, ST} is occasionally weakly compatible, we have

$$Qz = STz = z.$$

Now we show that Bz = z, by putting x = Bz and y = x_{2n+1} with $\beta = 1$ in (3.3.4) we have

$$M(PBz, Qx_{2n+1}, kt) \geq M(AB(Bz), P(Bz), t) * M(STx_{2n+1}, Qx_{2n+1}, t)$$

$$* M(STx_{2n+1}, PBz, t) * M(AB(Bz), Qx_{2n+1}, t)$$

$$* M(AB(Bz), STx_{2n+1}, t).$$

Proceeding limits as $n \to \infty$ and using Lemma 2.2, we have Bz = z. Since ABz = z, therefore, Pz = ABz = Bz = z = Qz = STz.

Finally, we show that Tz = z, by putting x = z and y = Tz with $\beta = 1$ in (3.3.4).

$$M(Pz, Q(Tz), kt) \geq M(ABz, Pz, t) * M(ST(Tz), Q(Tz), t)$$

$$* M(ST(Tz), Pz, t) * M(ABz, Q(Tz), t)$$

$$* M(ABz, ST(Tz), t).$$

Therefore, Tz = z.

Hence, ABz = Bz = STz = Tz = Pz = Qz = z.

Uniqueness follows easily.

If we put B = T = I, the identity map on X, in Theorem 3.3.1, we have the following:

Corollary 3.3.1. Let (X, M, *) be a complete fuzzy metric space with $t * t \geq t$ for all $t \in (0, 1)$ and let A, S, P and Q be the mapping from X into itself such that

(3.3.5) $P(X) \subset B(X)$, $Q(X) \subset S(X)$.

(3.3.6) The pair (P, A) is semi-compatible and (Q, S) is occasionally weakly compatible.

(3.3.7) There exists a number $k \in (0, 1)$ such that

$$M(Px, Qy, kt) \geq M(Ax, Px, t) * M(Sy, Qy, t) * M(Sy, Px, \beta t)$$

$$* M(Ax, Qy, (2 - \beta)t) * M(Ax, Sy, t);$$

for all $x, y \in X$, $\beta \in (0, 2)$ with $t > 0$.

If the range of one of the subspaces is complete then A, S, P and Q have a unique common fixed point in X.

Remark 3.2. Theorem 3.1 is a generalization of the result of Sharma [13] in the sense that condition of compatibility of type (A) of the pairs of self maps has been restricted to semi-compatible and occasionally weakly compatible self maps and continuity of the mappings have been completely removed.

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