FIXED POINTS IN NON-ARCHIMEDEAN MENGER PM-SPACE

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ABSTRACT

The concept of occasionally weakly compatible mappings is used to prove a common fixed point theorem. The theorem thus obtained is a generalization and extension of the result of Khan and Sumitra [13] in a non-Archimedean Menger PM-space.

Keywords: ***Non-Archimedean Menger Probabilistic Metric Space, Common Fixed Points, Compatible Maps, Occasionally Weakly Compatible Maps.***

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I. INTRODUCTION

The notion of non-Archimedean Menger space has been established by Istrătescu and Crivat [9]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istrătescu [6]. This has been the extension of the results of Sehgal and Bharucha - Reid [16] and Sherwood [17] on a Menger space. Cho et. al. [3] proved a common fixed point theorem for compatible mappings in non-Archimedean Menger PM-space. Achari [1] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istrătescu [7]. Recently Khan and Sumitra [13] proved a common fixed point theorem for three pointwise R-weakly commuting mappings in complete non-Archimedean Menger PM-spaces. In the present paper we prove a unique common fixed point theorem for four occasionally weakly compatible self maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the results of Khan and Sumitra [13] and others.

II. PRELIMINARIES

Definition 2.1. [7] Let X be a non-empty set and $\mathcal{D}$ be the set of all left-continuous distribution functions. An ordered pair (X, $\mathcal{F}$) is called a non-Archimedean probabilistic metric space (briefly, a N.A. PM-space) if $\mathcal{F}$ is a mapping from X×X into $\mathcal{D}$ satisfying the following conditions (the distribution function $\mathcal{F}(x,y)$ is denoted by $F_{x,y}$ for all $x,y \in X$) :

(PM-1) F(x, y; t) = 1, for all t > 0, if and only if u = v ;

(PM-2) F(x, y; t) = F(y, x; t);

(PM-3) F(x, y; 0) = 0;

(PM-4) If $F(x, y; t_1) = F(y, z; t_2) = 1$ then $F(x, z; \max\{t_1, t_2\}) = 1$,

for all $x, y, z \in X$.

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Definition 2.2. [14] A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, nondecreasing in each coordinate and $\Delta(a,1) = a$ for every $a \in [0,1]$.

Definition 2.3. [8] A *N.A. Menger PM-space* is an ordered triple $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a non-Archimedean PM-space and Δ is a t-norm satisfying the following condition:

(PM-5) $F_{u,w}(\max\{x,y\}) \geq \Delta(F_{u,v}(x), F_{v,w}(y))$,

for all $u, v, w \in X$ and $x, y \geq 0$.

Definition 2.4. [2] A PM-space $(X, \mathcal{F})$ is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that $g(F_{x,y}(t)) \leq g(F_{x,z}(t)) + g(F_{z,y}(t))$

for all $x, y, z \in X$ and $t \geq 0$, where $\Omega = \{g \mid g : [0,1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 2.5. [2] A N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that $g(\Delta(s,t)) \leq g(s) + g(t)$ for all $s, t \in [0,1]$.

Remark 2.1. [2]

(1) If a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$ then $(X, \mathcal{F}, \Delta)$ is of type $(C)_g$.

(2) If a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by

$$d(x,y) = \int_0^1 g(F_{x,y}(t))\,d(t) \text{ for all } x, y \in X. \qquad (*)$$

Throughout this paper, suppose $(X, \mathcal{F}, \Delta)$ be a complete N.A. Menger PM-space of type $(D)_g$ with a continuous strictly increasing t-norm Δ.

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a function satisfied the condition (Φ) :

(Φ) ϕ is upper-semicontinuous from the right and $\phi(t) < t$ for all $t > 0$.

Lemma 2.1. [3] If a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ), then we have

(1) For all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is n^{th} iteration of $\phi(t)$.

(2) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$ then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for all $t \geq 0$, then $t = 0$.

Definition 2.6. [10] Let $A, S : X \to X$ be mappings. A and S are said to be compatible if $\lim_{n\to\infty} g(F(ASx_n, SAx_n; t)) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

Definition 2.7. [11] Self maps A and S of a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ap = Sp$ for some $p \in X$ then
$ASp = SAp$.

Definition 2.8. Self maps A and S of a N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Lemma 2.2. [3] Let A, B, S, T : $X \to X$ be mappings satisfying the condition (1) and (2) as follows :

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

(2) $g(F(Ax,By; t)) \le \phi(\max\{g(F(Sx,Ty; t)), g(F(Sx, Ax; t)), g(F(Ty, By; t)),$
$\frac{1}{2}(g(F(Sx, By; t)) + g(F(Ty, Ax; t)))\})$

for all $t > 0$, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ). Then the sequence $\{y_n\}$ in X, defined by $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$, such that $\lim_{n\to\infty} g(F(y_n,y_{n+1}; t)) = 0$ for all $t > 0$ is a Cauchy sequence in X.

III. MAIN RESULT

Theorem 3.1. Let (X, F, Δ) be a complete N.A. Menger PM-space and A, B, S, T : $X \to X$ be mappings satisfying the conditions

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(3.2) the pairs (A, S) and (B, T) are occasionally weakly compatible and

(3.3) $g(F(Ax, By; t)) \le \phi[\max\{g(F(Sx, Ty; t)), g(F(Sx, Ax; t)),$
$g(F(Ty, By; t)), \frac{1}{2}(g(F(Sx, By; t)) + g(F(Ty, Ax; t)))\}]$

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subseteq T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_n = Ax_{2n} = Tx_{2n+1},\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 1, 2, \ldots . \quad (1)$$

Let $M_n = g(F(Ax_n, Bx_{n+1}; t) = g(F(y_n, y_{n+1}; t)$ for $n = 1, 2, \ldots$. Then

$$M_{2n} = g(F(Ax_{2n}, Bx_{2n+1}; t)$$
$$\le \phi[\max\{g(F(Sx_{2n}, Tx_{2n+1}; t)), g(F(Sx_{2n}, Ax_{2n}; t)), g(F(Tx_{2n+1}, Bx_{2n+1}; t)),$$
$$\frac{1}{2}(g(F(Sx_{2n}, Bx_{2n+1}; t)) + g(F(Tx_{2n+1}, Ax_{2n}; t)))\}]$$
$$\le \phi[\max\{g(F(y_{2n-1}, y_{2n}; t)), g(F(y_{2n-1}, y_{2n}; t)), g(F(y_{2n}, y_{2n+1}; t)),$$
$$\frac{1}{2}(g(F(y_{2n-1}, y_{2n+1}; t)) + g(F(y_{2n}, y_{2n}; t)))\}]$$

i.e. $M_{2n} \le \phi[\max\{M_{2n-1}, M_{2n-1}, M_{2n}, \frac{1}{2}(M_{2n-1} + M_{2n})\}]$. (2)

If $M_{2n} > M_{2n-1}$ then by (2),

$$M_{2n} \ge \phi(M_{2n}), \text{ a contradiction.}$$

If $M_{2n-1} > M_{2n}$ then by (2),

$$M_{2n} \le \phi(M_{2n-1}).$$

So by Lemma 2.1, we have $\lim_{n\to\infty} M_{2n} = 0$, i.e.

$$\lim_{n\to\infty} g(F(Ax_{2n}, Bx_{2n+1}; t)) = 0$$

i.e. $\lim_{n\to\infty} g(F(y_{2n}, y_{2n+1}; t)) = 0$.

Similarly, we can show that

$\lim_{n\to\infty} g(F(Bx_{2n+1}, Ax_{2n+2}; t)) = 0$

i.e. $\lim_{n\to\infty} g(F(y_{2n+1}, y_{2n+2}; t)) = 0.$

Thus, we have

$\lim_{n\to\infty} g(F(Ax_{2n}, Bx_{n+1}; t)) = 0$ for all $t > 0$

i.e. $\lim_{n\to\infty} g(F(y_n, y_{n+1}; t)) = 0$ for all $t > 0$. (3)

Hence, by Lemma 2.2, the sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, so the sequence $\{x_n\}$ converges to a point z in X and so the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ also converges to the limit z.

Since $B(X) \subseteq S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (3.3), we have

$g(F(Au, z; t)) \le g(F(Au, Bx_{2n-1}; t)) + g(F(Bx_{2n-1}, z; t))$

$\le \phi[\max\{g(F(Su, Tx_{2n-1}; t)), g(F(Su, Au; t)), g(F(Tx_{2n-1}, Bx_{2n-1}; t),$

$\tfrac{1}{2}(g(F(Su, Bx_{2n-1}; t)) + g(F(Tx_{2n-1}, Au; t)))\}].$

Letting $n \to \infty$, we get

$g(F(Au, z; t)) \le \phi[\max\{g(z, z; t)), g(F(z, Au; t)), g(F(z, z; t)), \tfrac{1}{2}(g(F(z, z; t)) + g(F(z, Au; t)))\}]$

$= \phi[\max\{0, g(F(z, Au; t)), 0, \tfrac{1}{2}(0 + g(F(z, Au; t)))\}]$

$\le \phi(g(F(Au, z; t)))$

for all $t > 0$, which implies that $g(F(Au, z; t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore, $Au = Su = z$. Since $A(X) \subseteq T(X)$, there exists a point v in X such that $z = Tv$. Again, using (3.3), we have

$g(F(z, Bv; t)) = g(F(Au, Bv; t))$

$\le \phi[\max\{g(F(Su, Tv; t)), g(F(Su, Au; t)), g(F(Tv, Bv; t)),$

$\tfrac{1}{2}(g(F(Su, Bv ; t)) + g(F(Tv, Au; t)))\}]$

$\le \phi[\max\{g(F(z, z; t)), g(F(z, z; t)), g(F(z, Bv; t)),$

$\tfrac{1}{2}(g(F(z, Bv ; t)) + g(F(z, z; t)))\}]$

$= \phi[\max\{0, 0, g(F(z, Bv; t)), \tfrac{1}{2}(g(F(z, Bv ; t)) + 0)\}]$

$\le \phi(g(F(Bv, z; t)))$ for all $t > 0$,

which implies that $g(F(Bv, z; t)) = 0$ for all $t > 0$ by Lemma 2.1.

Therefore, $Bv = Tv = z$. Since A and S are occasionally weakly compatible mappings, $ASz = SAz$ i.e. $Az = Sz$. Now we show that z is a fixed point of A. If $Az \ne z$, then by (3.3), we have

$g(F(Az, z; t)) = g(F(Az, Bv; t))$

$\le \phi[\max\{g(F(Sz, Tv; t)), g(F(Sz, Az; t)), g(F(Tv, Bv; t)),$

$\tfrac{1}{2}(g(F(Sz, Bv ; t)) + g(F(Tv, Az; t)))\}]$

$\le \phi[\max\{g(F(Az, z; t)), 0, 0, \tfrac{1}{2}(g(F(Az, z; t)) + g(F(z, Az; t)))\}]$

$\le \phi(g(F(Az, z; t)))$ for all $t > 0$,

which implies that $g(F(Az, z; t) = 0$ for all $t > 0$ by Lemma 2.1. Therefore, $Az = z$.

Hence, $Az = Sz = z$.

Similarly, as B and T are occasionally weakly compatible mappings, we have

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$Bz = Tz = z$, since by (3.3), we have

$g(F(z, Bz; t)) = g(F(Az, Bz; t))$

$\leq \phi[\max\{g(F(Sz, Tz; t)), g(F(Sz, Az; t)), g(F(Tz, Bz; t)),$

$\frac{1}{2}(g(F(Sz, Bz ; t)) + g(F(Tz, Az; t)))\}]$

$\leq \phi[\max\{g(F(z, Bz; t)), 0, 0, \frac{1}{2}(g(F(z, Bz; t)) + g(F(Bz, z; t)))\}]$

$\leq \phi(g(F(Bz, z; t)))$ for all $t > 0$,

which implies that $g(F(Bz, z; t) = 0$ for all $t > 0$ by Lemma 2.1. Therefore, $Bz = z$.

Hence, $Bz = Tz = z$.

Thus, $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose that w is another common fixed point of A, B, S and T. Then by (3.3), we have

$g(F(z, w; t)) = g(F(Az, Bw; t))$

$\leq \phi[\max\{g(F(Sz, Tw; t)), g(F(Sz, Az; t)), g(F(Tw, Bw; t)),$

$\frac{1}{2}(g(F(Sz, Bw; t)) + g(F(Tz, Aw; t)))\}]$

$\leq \phi(g(F(z, w; t))$ for all $t > 0$,

which implies that $g(F(z, w; t)) = 0$ for all $t > 0$ by Lemma 2.1.

Hence, $z = w$.

Therefore, z is a unique common fixed point of A, B, S and T.

Corollary 3.1. Let $A, S, T : X \to X$ be the mappings satisfying

(i) $A(X) \subseteq S(X) \cap T(X)$,

(ii) the pairs $\{A, S\}$ and $\{A, T\}$ are occasionally weakly compatible and

(iii) $g(F(Ax, Ay; t)) \leq \phi[\max\{g(F(Sx, Ty; t)), g(F(Sx, Ax; t)), g(F(Ty, Ay; t)),$
$\frac{1}{2}(g(F(Sx, Ay; t)) + g(F(Ty, Ax; t)))\}]$,

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, S and T have a unique common fixed point in X.

Corollary 3.2. Let $A, S : X \to X$ be the mappings satisfying

(i) $A(X) \subseteq S(X)$,

(ii) the pairs $\{A, S\}$ is occasionally weakly compatible and

(iii) $g(F(Ax, Ay; t)) \leq \phi[\max\{g(F(Sx, Sy; t)), g(F(Sx, Ax; t)),$
$g(F(Sy, Ay; t)), \frac{1}{2}(g(F(Sx, Ay; t)) + g(F(Sy, Ax; t)))\}]$,

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A and S have a unique common fixed point in X.

Remark 3.1. In Theorem 3.1, if S and T are continuous and pairs $\{A, S\}$ and $\{B, T\}$ are compatible instead of condition (3.2), the theorem remains true.

Remark 3.2. In our generalization the inequality condition (3.3) satisfied by the mappings A, B, S and T is stronger than that of Theorem 2 of Khan and Sumitra [13] and Theorem 1.9 of Vasuki [20].

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