ON SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY HYPERBOLIC COSYMPLECTIC MANIFOLD WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

Toukeer Khan[1], Shadab Ahmad Khan[2], Mobin Ahmad[3]

[1]*Department of Mathematics, Integral University, Lucknow (India)*

[2]*Assistant Professor, Department of Mathematics, Integral University, Lucknow (India)*

[3]*Associate Professor, Department of Mathematics, Jazan University, Jazan (Saudi Arabia)*

ABSTRACT

We consider a nearly hyperbolic cosymplectic manifold and study semi-invariant sub manifolds of a nearly hyperbolic cosymplectic manifold admitting semi-symmetric non-metric connection. We also find the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold with semi-symmetric non-metric connection and study parallel distributions on them.

Keywords: Integrability Condition, Nearly Hyperbolic Cosymplectic Manifold, Parallel Distribution, Semi-Invariant Sub Manifolds, Semi-Symmetric Non-Metric Connection.

2000 AMS Mathematics Subject Classification: 53D05, 53D25, 53D12.

I. INTRODUCTION

In 1978, A. Bejancu [1] initiated the concept of CR- sub manifolds of a Kaehler manifold as generalization of invariant and anti-invariant sub manifolds. The extension of the concept of a CR-sub manifold of Kaehler manifold is a semi-invariant sub manifold to sub manifolds of an almost contact manifold. A semi-invariant sub manifold of a Sasakian manifold was initially studied by Bejancu - Papaghuic [2]. In 1983, K. Matsumoto [3] and Yano-Kon [4] studied the same concept under the name of contact CR-sub manifold. The study of semi-invariant sub manifolds in almost contact manifold was enriched by several geometers (see, [5], [6], [7], [8], [9], [10]). On the otherhand, Golab [11] introduced the idea of semi-symmetric and quarter-symmetric connection. Upadhyay and Dube [12] studied and define the almost hyperbolic (f, g, η, ξ)-structure. A semi-invariant sub manifolds of an almost r-contact hyperbolic metric manifolds was studied by Joshi and Dube [13]. Ahmad M. and Ali K., studied semi-invariant sub manifolds of a nearly hyperbolic cosymplectic manifold in [14].

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\bar{M}$. The torsion tensor T and curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. A.

Friedmann and J. A. Schouten [15] introduced the idea of a semi- symmetric connection. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form:

$T(X,Y) = \eta(X)Y - \eta(Y)X$

Many geometers (see, [16], [17]) have studied properties of semi-symmetric non- metric connection.

In this paper, we study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold with semi-symmetric non-metric connection.

This paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold. In section 3, we study some properties of semi invariant submanifolds of a nearly hyperbolic cosymplectic manifold with semi-symmetric non-metric connection. We also study parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold with semi-symmetric non-metric connection. In section 4, we discuss the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold with semi-symmetric non-metric connection.

II. PRELIMINARIES

Let $\bar{M}$ be an n-dimensional almost hyperbolic Contact metric manifold with the almost hyperbolic contact metric structure $(\emptyset,\xi,\eta,g)$, where $\emptyset$ is a tensor of type $(1,1)$, ξ is a vector field called structure vector field and η is the dual 1-form of ξ and the associated Riemannian metric g satisfying the following

$$\emptyset^2 X = X + \eta(X)\xi \quad (2.1)$$

$$\eta(\xi) = -1, \quad g(X,\xi) = \eta(X) \quad (2.2)$$

$$\emptyset(\xi) = 0, \quad \eta o \emptyset = 0 \quad (2.3)$$

$$g(\emptyset X, \emptyset Y) = -g(X,Y) - \eta(X)\eta(Y) \quad (2.4)$$

for any X, Y tangent to $\bar{M}$ [8]. In this case

$$g(\emptyset X, Y) = -g(\emptyset Y, X) \quad (2.5)$$

An almost hyperbolic contact metric structure $(\emptyset,\xi,\eta,g)$ on $\bar{M}$ is called nearly hyperbolic cosymplectic manifold [8] if and only if

$$(\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X = 0 \quad (2.6)$$

$$\nabla_X \xi = 0 \quad (2.7)$$

for all X, Y tangent to $\bar{M}$, where ∇ is Riemannian connection $\bar{M}$.

Now, we define a semi-symmetric non-metric connection

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \quad (2.8)$$

such that $(\bar{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y)$

From (2.6) & (2.8), replacing Y by $\emptyset Y$, we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -\eta(X)\emptyset Y - \eta(Y)\emptyset X \quad (2.9)$$

$$\bar{\nabla}_X \xi = -X \quad (2.10)$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure $(\emptyset,\xi,\eta,g)$ is called nearly hyperbolic Cosymplectic manifold with semi-symmetric non-metric connection if it is satisfied (2.9) & (2.10).

III. SEMI-INVARIANT SUBMANIFOLDS AND SOME BASIC RESULTS

Let M be submanifold immersed in $\bar{M}$, we assume that the vector ξ is tangent to M, denoted by $\{\xi\}$ the 1-dimentional distribution spanned by ξ on M, then M is called a semi-invariant submanifold [7] of $\bar{M}$ if there exist two differentiable distribution D & $D^{\perp}$ on M satisfying

(i) $TM = D \oplus D^{\perp} \oplus \xi$, where D , $D^{\perp}$ & ξ are mutually orthogonal to each other.

(ii) The distribution D is invariant under $\emptyset$, i.e. $\emptyset D_X = D_X$ for each $X \in M$,

(iii) The distribution $D^{\perp}$ is anti-invariant under $\emptyset$, i.e. $\emptyset {D^{\perp}}_X \subset T^{\perp}M$ for each $X \in M$, where TM & $T^{\perp}M$ be the Lie algebra of vector fields tangential & normal to M respectively.

Let Riemannian metric g and ∇ be induced Levi-Civita connection on M then the Guass formula & Weingarten formula are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{3.2}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form & A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X,Y), N) \tag{3.3}$$

Any vector X tangent to M is given as

$$X = PX + QX + \eta(X)\xi \tag{3.4}$$

where $PX \in D \; and \; QX \in D^{\perp}$.

Similarly, for N normal to M, we have

$$\emptyset N = BN + CN \tag{3.5}$$

where BN $(resp. CN)$ is tangential component $(resp. normal \; component)$ of $\emptyset N$.

Using the semi-symmetric non-metric connection the Nijenhuis tensor is expressed as

$$N(X,Y) = (\bar{\nabla}_{\emptyset X}\emptyset)Y - (\bar{\nabla}_{\emptyset Y}\emptyset)X - \emptyset(\bar{\nabla}_X\emptyset)Y + \emptyset(\bar{\nabla}_Y\emptyset)X \tag{3.6}$$

Now from (2.9) replacing$X \; by \; \emptyset X$, we have

$$(\bar{\nabla}_{\emptyset X}\emptyset)Y = -\eta(Y)X - \eta(X)\eta(Y)\xi - (\bar{\nabla}_Y\emptyset)\emptyset X \tag{3.7}$$

Differentiating (2.1) conveniently along the vector and using (2.10), we have

$$(\bar{\nabla}_Y\emptyset)\emptyset X = (\bar{\nabla}_Y\eta)(X)\xi - \eta(X)Y - \emptyset(\bar{\nabla}_Y\emptyset)X \tag{3.8}$$

From (3.7) & (3.8), we have

$$(\bar{\nabla}_{\emptyset X}\emptyset)Y = \eta(X)Y - \eta(Y)X - (\bar{\nabla}_Y\eta)(X)\xi - \eta(X)\eta(Y)\xi + \emptyset(\bar{\nabla}_Y\emptyset)X \tag{3.9}$$

Interchanging X & Y, we have

$$(\bar{\nabla}_{\emptyset Y}\emptyset)X = \eta(Y)X - \eta(X)Y - (\bar{\nabla}_X\eta)(Y)\xi - \eta(X)\eta(Y)\xi + \emptyset(\bar{\nabla}_X\emptyset)Y \tag{3.10}$$

Using equation (3.9), (3.10) and (2.9) in (3.6), we have

$$N(X,Y) = 4\eta(X)Y + 2g(\emptyset X, Y)\xi + 4\eta(X)\eta(Y)\xi + 4\emptyset(\bar{\nabla}_Y\emptyset)X \tag{3.11}$$

As we know, $(\bar{\nabla}_Y\emptyset)X = \bar{\nabla}_Y\emptyset X - \emptyset(\bar{\nabla}_Y X)$

Using Guass formula (3.1), we have

$$\emptyset(\bar{\nabla}_Y\emptyset)X = \emptyset(\nabla_Y\emptyset X) + \emptyset h(Y,\emptyset X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(Y,X) \tag{3.12}$$

Using equation (3.12) in (3.11), we have

$$N(X,Y) = 4\eta(X)Y + 4\eta(X)\eta(Y)\xi + 4\emptyset(\nabla_Y \emptyset X) + 4\emptyset h(Y,\emptyset X) - 4(\nabla_Y X)$$
$$-4\eta(\nabla_Y X)\xi - 4h(Y,X) + 2g(\emptyset X,Y)\xi \quad (3.13)$$

Lemma 3.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$2(\bar{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

for all $X, Y \in D$.

Proof. By Gauss formula (3.1), we have

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) \quad (3.14)$$

Also, by covariant differentiation, we know that

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = (\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset[X,Y] \quad (3.15)$$

From (3.14) and (3.15), we have

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y] \quad (3.16)$$

Adding (2.9) and (3.16), we obtain

$$2(\bar{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

for all $X, Y \in D$.

Hence lemma is proved.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$2(\bar{\nabla}_Y \emptyset)X = \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + \emptyset[X,Y]$$

for all $X, Y \in D$.

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$2(\bar{\nabla}_X \emptyset)Y = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.

Proof. Using Weingarten formula (3.2), we have

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset X - \nabla_Y^{\perp} \emptyset X \quad (3.17)$$

Comparing equation (3.15) & (3.17), we have

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X,Y] \quad (3.18)$$

Adding (2.9) & (3.18), we have

$$2(\bar{\nabla}_X \emptyset)Y = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.

Hence lemma is proved.

Lemma 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$2(\bar{\nabla}_Y \emptyset)X = A_{\emptyset Y} X - A_{\emptyset X} Y + \nabla_Y^{\perp} \emptyset X - \nabla_X^{\perp} \emptyset Y + \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.

Lemma 3.5. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$2(\bar{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y]$$

for all $X \in D \, and \, Y \in D^{\perp}$.

Proof. By Gauss formulas (3.1) and Weingarten formula (3.2), we have

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) \quad (3.19)$$

Comparing equation (3.15) and (3.19), we have

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y] \quad (3.20)$$

Adding equation (2.9) & (3.20), we get

$$2(\bar{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y]$$

for all $X \in D \, and \, Y \in D^{\perp}$.

Hence lemma is proved.

Lemma 3.6. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$2(\bar{\nabla}_Y \emptyset)X = A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y \emptyset X + h(Y, \emptyset X) + \emptyset[X, Y]$$

for all $X \in D \, and \, Y \in D^{\perp}$.

Lemma 3.7. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$P(\nabla_X \emptyset PY) + P(\nabla_Y \emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y = -\eta(X)\emptyset PY - \eta(Y)\emptyset PX + \emptyset P(\nabla_X Y)$$
$$+ \emptyset P(\nabla_Y X) \quad (3.21)$$

$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y = 2Bh(X, Y) \quad (3.22)$$

$$h(X, \emptyset PY) + h(Y, \emptyset PX) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX = -\eta(X)\emptyset QY - \eta(Y)\emptyset QX + \emptyset Q(\nabla_X Y)$$
$$+ \emptyset Q(\nabla_Y X) + 2Ch(X, Y) \quad (3.23)$$

$$\eta(\nabla_X \emptyset PY) + \eta(\nabla_Y \emptyset PX) - \eta(A_{\emptyset QY}X) - \eta(A_{\emptyset QX}Y) = 0 \quad (3.24)$$

for all $X, Y \in TM$.

Proof. Differentiating covariantly equation (3.4) and using equation (3.1) and (3.2), we have

$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X, Y) = \nabla_X \emptyset PY + h(X, \emptyset PY) - A_{\emptyset QY}X + \nabla_X^{\perp}\emptyset QY \quad (3.25)$$

Interchanging X & Y, we have

$$(\bar{\nabla}_Y \emptyset)X + \emptyset(\nabla_Y X) + \emptyset h(Y, X) = \nabla_Y \emptyset PX + h(Y, \emptyset PX) - A_{\emptyset QX}Y + \nabla_Y^{\perp}\emptyset QX \quad (3.26)$$

Adding equations (3.25) & (3.26), we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X, Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX + h(X, \emptyset PY)$$
$$+ h(Y, \emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX \quad (3.27)$$

By Virtue of (2.9) & (3.27), we have

$$-\eta(X)\emptyset Y - \eta(Y)\emptyset X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X, Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX + h(X, \emptyset PY)$$
$$+ h(Y, \emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX$$

Using equations (3.4), (3.5) & (2.3), we have

$$-\eta(X)\phi PY - \eta(X)\phi QY - \eta(Y)\phi PX - \eta(Y)\phi QX + \phi P(\nabla_X Y) + \phi Q(\nabla_X Y) + \phi P(\nabla_Y X)$$

$$+\phi Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(X,Y) = P(\nabla_X \phi PY) + Q(\nabla_X \phi PY) + \eta(\nabla_X \phi PY)\xi$$

$$+P(\nabla_Y \phi PX) + Q(\nabla_Y \phi PX) + \eta(\nabla_Y \phi PX)\xi + h(X,\phi PY) + h(Y,\phi PX) - PA_{\phi QY}X$$

$$-QA_{\phi QY}X - \eta(A_{\phi QY}X)\xi - PA_{\phi QX}Y - QA_{\phi QX}Y - \eta(A_{\phi QX}Y)\xi + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX$$

Comparing horizontal, vertical and normal components we get desired results.

Hence lemma is proved.

Definition 3.8. The horizontal distribution D is said to be parallel [2] on M if $\nabla_X Y \in D$, for all $X, Y \in D$

Theorem 3.9. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection. If horizontal distribution D is parallel, then

$$h(X, \phi Y) = h(Y, \phi X)$$

for all $X, Y \in D$.

Proof. Let $X, Y \in D$, as D is parallel distribution, then

$$\nabla_X \phi Y \in D \;\&\; \nabla_Y \phi X \in D$$

Then, from (3.22) and (3.23), we have

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y + h(X,\phi PY) + h(Y,\phi PX) + \nabla_X^{\perp}\phi QY$$

$$+\nabla_Y^{\perp}\phi QX = -\eta(X)\phi QY - \eta(Y)\phi QX + \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(X,Y)$$

As Q being a projection operator on $D^{\perp}$ then we have

$$h(X,\phi Y) + h(Y,\phi X) = 2\phi h(X,Y) \quad (3.28)$$

Replacing X by ϕX in (3.28) & using (2.1), we have

$$h(\phi X,\phi Y) + h(Y,X) = 2\phi h(\phi X,Y) \quad (3.29)$$

Replacing Y by ϕY & using (2.1) in (3.28), we have

$$h(X,Y) + h(\phi Y,\phi X) = 2\phi h(X,\phi Y) \quad (3.30)$$

By Virtue of (3.29) and (3.30), we have

$$h(X,\phi Y) = h(Y,\phi X)$$

for all $X, Y \in D$.

Hence theorem is proved.

Definition 3.10. A semi-invariant submanifold is said to be mixed totally geodesic [2] if $h(X,Y) = 0,$ $for\ all\ X \in D\ and\ Y \in D^{\perp}$.

Theorem 3.11. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection. Then M is a mixed totally geodesic if and only if

$$A_N X \in D \qquad \text{for all } X \in D.$$

Proof. Let $A_N X \in D$ $for\ all\ X \in D$.

Now, $g(h(X,Y),N) = g(A_N X, Y) = 0,$ for $Y \in D^{\perp}$.

Which is equivalent to $h(X,Y) = 0$.

Hence M is totally mixed geodesic.

Conversely, Let M is totally mixed geodesic.

That is $h(X,Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$.

Now, $g(h(X,Y),N) = g(A_N X, Y)$.

This implies that $g(A_N X, Y) = 0$

Consequently, we have

$A_N X \in D$, for all $Y \in D^{\perp}$.

Hence theorem is proved.

IV. INTEGRABILITY OF DISTRIBUTION

Theorem 4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then the distribution $D \oplus \langle \xi \rangle$ is integrable if

$$h(X, \emptyset Z) = h(\emptyset X, Z) \tag{4.1}$$

for each $X, Y, Z \in (D \oplus \langle \xi \rangle)$.

Proof. The torsion tensor $S(X,Y)$ of an almost hyperbolic contact manifold is given by

$$S(X,Y) = N(X,Y) + 2d\eta(X,Y)\xi$$

Where $N(X,Y)$ is Neijenhuis tensor.

If $(D \oplus \langle \xi \rangle)$ is integrable,

then $N(X,Y) = 0$, *for any* $X, Y \in (D \oplus \langle \xi \rangle)$

Hence from (3.13), we have

$$4\eta(X)Y + 4\eta(X)\eta(Y)\xi + 4\emptyset(\nabla_Y \emptyset X) + 4\emptyset h(Y, \emptyset X) - 4(\nabla_Y X) - 4\eta(\nabla_Y X)\xi - 4h(Y,X)$$
$$+2g(\emptyset X, Y)\xi = 0 \tag{4.2}$$

Comparing normal part both side of (4.2), we have

$$\emptyset Q(\nabla_Y \emptyset X) - h(Y,X) + Ch(Y, \emptyset X) = 0 \tag{4.3}$$

for $X, Y \in (D \oplus \langle \xi \rangle)$

Replacing Y *by* $\emptyset Z$, *where* $Z \in D$ in (4.3), we have

$$\emptyset Q(\nabla_{\emptyset Z} \emptyset X) - h(\emptyset Z, X) + Ch(\emptyset Z, \emptyset X) = 0 \tag{4.4}$$

Interchanging X *and* Z, we have

$$\emptyset Q(\nabla_{\emptyset X} \emptyset Z) - h(\emptyset X, Z) + Ch(\emptyset X, \emptyset Z) = 0 \tag{4.5}$$

Subtracting (4.4) from (4.5), we obtain

$$\emptyset Q[\emptyset X, \emptyset Z] - h(\emptyset X, Z) + h(\emptyset Z, X) = 0 \tag{4.6}$$

Since $(D \oplus \langle \xi \rangle)$ is integrable,

So that $[\emptyset X, \emptyset Z] \in (D \oplus \langle \xi \rangle)$, for $X, Z \in D$

Consequently, (4.6) gives

$$h(\emptyset X, Z) = h(\emptyset Z, X)$$

for each $X, Y, Z \in (D \oplus \langle \xi \rangle)$.

Hence theorem is proved.

Theorem 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$A_{\emptyset Y}Z - A_{\emptyset Z}Y = \frac{1}{3}\emptyset P[Y,Z]$$

for each $Y, Z \in D^{\perp}$.

Proof. Let $Y, Z \in D^{\perp}$ and $X \in TM$, from (3.3), we have

$$2g(A_{\emptyset Z}Y, X) = g(h(Y,X), \emptyset Z) + g(h(X,Y), \emptyset Z) \tag{4.7}$$

Using (2.9) & (3.1) in (4.7), we have

$$2g(A_{\emptyset Z}Y, X) = -g(\nabla_Y \emptyset X, Z) - g(\nabla_X \emptyset Y, Z) - \eta(X)g(\emptyset Y, Z) - \eta(Y)g(\emptyset X, Z) \tag{4.8}$$

From (3.2), we have

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

Replacing N by $\emptyset Y$

$$\bar{\nabla}_X \emptyset Y = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y$$

As ∇ is a Levi-Civita connection, using above, then from (4.8), we have

$$2g(A_{\emptyset Z}Y, X) = -g(\emptyset \nabla_Y Z, X) + g(A_{\emptyset Y}Z, X) \tag{4.9}$$

Transvecting X from both sides from (4.9), we obtain

$$2A_{\emptyset Z}Y = -\emptyset \nabla_Y Z + A_{\emptyset Y}Z \tag{4.10}$$

Interchanging Y & Z, we have

$$2A_{\emptyset Y}Z = -\emptyset \nabla_Z Y + A_{\emptyset Z}Y \tag{4.11}$$

Subtracting (4.10) from (4.11), we have

$$(A_{\emptyset Y}Z - A_{\emptyset Z}Y) = \frac{1}{3}\emptyset[Y,Z]$$

Comparing the tangential part both side in above equation, we have

$$(A_{\emptyset Y}Z - A_{\emptyset Z}Y) = \frac{1}{3}\emptyset P[Y,Z]$$

where $[Y,Z]$ is Lie Bracket.

Hence theorem is proved.

Theorem 4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then the distribution is Integrable if and only if

$$A_{\emptyset Y}Z - A_{\emptyset Z}Y = 0 \tag{4.12}$$

for all $Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable, that is $[Y,Z] \in D^{\perp}$

For any $Y, Z \in D^{\perp}$, therefore $P[Y,Z] = 0$.

Consequently, from (4.11) we have

$$A_{\emptyset Y}Z - A_{\emptyset Z}Y = 0$$

Conversely, let (4.12) holds. Then by virtue of (4.11), we have

$$\emptyset P[Y,Z] = 0$$

$For\ all\ Y, Z \in D^{\perp}$. Since $rank\ \emptyset = 2n$

Therefore, either $P[Y,Z] = 0$ or $P[Y,Z] = k\xi$.

But $P[Y,Z] = k\xi$ is not possible as P being a projection operator on D.

So $$P[Y,Z] = 0,$$

This implies that $[Y,Z] \in D^{\perp}$, for all $Y,Z \in D^{\perp}$.

Hence $D^{\perp}$ is integrable.

Hence theorem is proved.

REFERENCES

[1] BEJANCU, A., 'CR- submanifolds of a Kaehler manifold I', *Proc. Amer. Math. Soc.,* (1978), *pp 135-142.*

[2] BEJANCU, A. and PAPAGHUIC, N., 'Semi-invariant submanifolds of a Sasakian manifold', *An. St. Univ. Al. I. Cuza, Iasi 27,* (1981), *pp 163-170.*

[3] MATSUMOTO, K., 'On contact CR-submanifolds of Sasakian manifold', *Intern. J. Math. Sci.,* (1983), *pp 313-326.*

[4] YANO, K. and KON, M., 'Contact -CR submanifolds', *Kodai Math. J. 5,* (1982), *pp 238-252.*

[5] AHMAD, M., 'Semi-invariant submanifolds of nearly Kenmotsu manifold with semi-symmetric semi-metric connection', *Mathematicki Vesnik 62,* (2010), *pp 189-198.*

[6] AHMAD, M. and JUN, J.B., 'On Semi-invariant submanifold of nearly Kenmotsu manifold with semi-symmetric non-metric connection', *Journal of the Chungcheong Math. Soc. Vol. 23, No. 2,* (2010), *pp 257-266.*

[7] AHMAD, M., KHAN, S.A. and KHAN, T., 'On semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with semi-symmetric semi-metric connection', *IOSR-JM, Vol. 10, Iss. 4 (Ver. V),* (2014), *pp 45-50.*

[8] BLAIR, D.E., 'Contact manifolds in Riemannian geometry', *Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin,* (1976).

[9] KHAN, T., KHAN, S.A. and AHAMD, M., 'On semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with semi-symmetric metric connection', *Int. Journal of Engineering Research and Application, Vol. 4, Iss. 9 (Ver. 3),* (2014), *pp 61-69.*

[10] SHAHID, M.H., 'On semi-invariant submanifolds of a nearly Sasakian manifold', *Indian J. Pure and Applied Math. 95 (10),* (1993), *pp 571-580.*

[11] GOLAB, S., 'On semi-symmetric and quarter symmetric linear connections', *Tensor, N.S., Japan,* (1975), *pp 249-254.*

[12] UPADHYAY, M. D. and DUBE, K.K., 'Almost contact hyperbolic (f,ξ,η,g)-structure', *Acta. Math. Acad. Scient. Hung. Tomus 28,* (1976), *pp 1-4.*

[13] JOSHI, N.K. and DUBE, K.K., 'Semi-invariant submanifolds of an almost r-contact hyperbolic metric manifold', *Demonstratio Mathematica, Poland,* (2001), *pp 135.143.*

[14] AHMAD, M. and ALI, K., 'Semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold', *Global Journal of Science Frontier Research Mathematics and Decision Sciences Journal of Mathematics, Vol. 13, Iss. 4, Ver. 1,* (2013), *pp 73-81.*

[15] FRIENDMANN, A. and SCHOUTEN, J.A., 'Uber die Geometric der halbsymmetrischen', *Ubertragung Math. Z., Springer-Verlag, Berlin, Germany,* (1924), *pp 211-223.*

[16] AHMAD, M. and JUN, J.B., 'On Semi-invariant submanifold of nearly Kenmotsu manifold with semi-symmetric non-metric connection', *Journal of the Chungcheong Math. Soc. Vol. 23, No. 2,* (2010), *pp 257-266.*

[17] DAS, LOVEJOY S., AHMAD, M. and HASEEB, A., 'Semi-invariant submanifolds of a nearly Sasakian manifold endowed with a semi-symmetric non-metric connection', *Journal of Applied Analysis, Vol. 17, No. 1,* (2011), *pp 119-130.*

Biographical Notes

Mr. Toukeer Khan is working as a Lecturer in Department of Mathematics, Integral University, Lucknow and presently pursuing in Ph.D from Integral University, Lucknow, India.

Dr. Shadab Ahmad khan is working as Assistant Professor in Department of Mathematics, Integral University, Lucknow, India.

Dr. Mobin Ahmad is working as Associate Professor in Department of Mathematics, Faculty of Science, Jazan University, Jazan-2069, Saudi Arabia.