SOME SUFFICIENT CONDITIONS FOR POISSON DISTRIBUTION SERIES ASSOCIATED WITH CONIC REGIONS

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ABSTRACT

The purpose of the present paper is to investigate some sufficient conditions for the convolution operator $I(m)f(z)$ belonging to the classes $k-UCV(\alpha)$, $k-S_p(\alpha)$, $S^*(\lambda)$ and $C(\lambda)$.

***Keywords:* Convex Functions, Harmonic Functions, Poisson Distribution Series, Starlike Function, Univalent Functions.**

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I. INTRODUCTION

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f^0(0) - 1 = 0$. Further, we denote by S the subclass of A consisting of functions of the form (1.1) which are also univalent in U. A function f of the form (1.1) is said to be strlike of order α if it satisfies the following condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in U.$$

and is said to be convex of order α if it satisfies the following condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in U.$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $C(\alpha)$, respectively, studied by Robertson [16], (see also [19]).

Bharti *et al.* [1] introduced the subclasses of k-uniformly convex functions of order α and corresponding class of starlike functions as follows

If $f \in A$, $0 \le k < \infty$ and $0 \le \alpha < 1$ then $f \in k - UCV(\alpha)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha. \qquad (1.2)$$

For $\alpha = 0$ the class $k - UCV(\alpha)$ reduce to the class $k - UCV$ introduced and studied by Kanas and Wisniowska [7] and for $k = 1, \alpha = 0$ it reduce to the class uniformly convex functions UCV studied by Goodman [4]. Using the Alexander transform we can obtain the class $k-S_p(\alpha)$ in the following way $f \in k-UCV(\alpha) \Leftrightarrow zf^0 \in k-S_p(\alpha)$.

For more results on these directions we refer the reader to ([2], [5], [6], [8], [9], [17], [20]) and references therein. A function $f \in A$ is said to be in the class $P_\gamma^\tau(\beta)$ if it satisfies the following inequality

$$\left|\frac{(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1}{2\tau(1-\beta)+(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1}\right|<1,$$

where $0 \le \gamma < 1$, $\beta < 1$, $\tau \in C/\{0\}$ and $z \in U$. The class $P_\gamma^\tau(\beta)$ was introduced by

Swaminathan [21].

Next, we introduce the classes S_λ^* and C_λ as follows

$$S_\lambda^* = \left\{f \in A : \left|\frac{zf'(z)}{f(z)}-1\right| < \lambda, \quad (z \in U, \lambda > 0)\right\} \qquad (1.3),(1.4)$$

$$C_\lambda = \left\{f \in A : \left|\frac{zf''(z)}{f'(z)}\right| < \lambda, \quad (z \in U, \lambda > 0)\right\}.$$

From (1.3) and (1.4) it is easy to see that

$$f(z) \in C_\lambda \Leftrightarrow zf'(z) \in S_\lambda^*, \quad (\lambda > 0).$$

The classes S_λ^* and C_λ were introduced by Ponnusamy and Rønning [10].

Very recently, Porwal [12] introduce a power series whose coefficients are probabilities of Poisson distribution

$$K(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$

By ratio test the radius of convergence of above series is infinity. Using the above series they obtain some interesting results on certain classes of analytic univalent functions.

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and

$g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Now, we consider the linear operator $I(m) : A \to A$ defined by

$$I(m)f = K(m,z) * f(z)$$
$$= z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n.$$

In the present paper, motivated by results of [12] and on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [3], [10], [14] , [18]) and by work of ([11], [13], [15]) we establish some sufficient conditions for convolution operator $I(m)f(z)$ belonging to the classes $k-UCV(\alpha)$, $k-S_p(\alpha)$, C_λ and S_λ^*.

II. MAIN RESULTS

To establish our main results, we shall require the following lemmas.

Lemma 2.1. ([1]) A function $f \in A$ is in $k-UCV(\alpha)$ if it satisfies the following condition

$$\sum_{n=2}^{\infty} n[n(1+k)-(k+\alpha)]|a_n| \le 1-\alpha. \qquad (2.1)$$

Remark 1. It was also found that the condition (2.1) is necessary if $f \in A$ is of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \ge 0. \qquad (2.2)$$

Lemma 2.2. ([1]) A function $f \in A$ is in $k - S_p(\alpha)$ if it satisfies the following inequality

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\alpha)]|a_n| \le 1-\alpha. \qquad (2.3)$$

The condition (2.3) is necessary for functions of the form (2.2).

Another sufficient condition is also given for the class $k - UCV\,(\alpha)$ in [8] which is given by the following way.

Lemma 2.3. ([7]) Let $f \in S$ and have the form (1.1). If for some k, $0 \le k < \infty$, the inequality

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \frac{1}{(k+2)},$$

holds, then $f \in k - UCV$. The number $1/k + 2$ can not be increased.

Lemma 2.4. Let $f \in A$ be of the form (1.1). If

$$\sum_{n=2}^{\infty}(\lambda+n-1)|a_n| \le \lambda, \quad (\lambda > 0) \quad , \qquad (2.4)$$

then $f \in S_\lambda^*$.

We further note that when $f(z)$ is of the form (2.2), the condition (2.4) is both necessary and sufficient for $f \in S_\lambda^*$.

Lemma 2.5. Let $f \in A$ be of the form (1.1). If

$$\sum_{n=2}^{\infty} n(\lambda+n-1)|a_n| \le \lambda, \quad (\lambda > 0), \qquad (2.5)$$

then $f \in C_\lambda$.

Lemma 2.6. If $f \in P_\gamma^\tau(\beta)$ is of the form (1.1) then

$$|a_n| \le \frac{2|\tau|(1-\beta)}{1+\gamma(n-1)}.$$

Theorem 2.1. Let $f \in A$ be defined as in (1.1) suppose that $m > 0$, $k \ge 0$, $0 \le \alpha < 1$ and the inequality

$$(k+1)m+(1-\alpha)\ \ 1-e^{-m}) \le \frac{\gamma(1-\alpha)}{2|\tau|(1-\beta)}$$

is satisfied then for $f \in P_\gamma^\tau(\beta), 0 < \gamma \le 1$ and $0 \le \beta < 1$, $I(m)f(z) \in k - UCV\,(\alpha)$.

Proof. Since

$$I(m)f(z) = z + \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_n z^n.$$

To prove that $I(m)f(z) \in k - UCV\,(\alpha)$, from Lemma 2.1, it is sufficient to show that

$$\sum_{n=2}^{\infty} n[n(1+k)-(k+\alpha)]|A_n| \le 1-\alpha, \qquad (2.6)$$

where

$$A_n = \frac{m^{n-1}}{(n-1)!}e^{-m}a_n, \quad n \ge 2.$$

Now

$$\sum_{n=2}^{\infty} n[n(1+k)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|$$

$$\le 2|\tau|(1-\beta)\sum_{n=2}^{\infty} n[n(1+k)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}\frac{1}{1+\gamma(n-1)}, \quad \text{(using Lemma 2.6)}$$

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$$\leq 2|\tau|(1-\beta)e^{-m}\sum_{n=2}[n(1+k)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}\frac{1}{\gamma}, \qquad (\text{Since } 1+\gamma(n-1)\geq \gamma n)$$

$$= \frac{2|\tau|(1-\beta)}{\gamma}e^{-m}\left[\sum_{n=2}^{\infty}[(n-1)(1+k)+(1-\alpha)]\frac{m^{n-1}}{(n-1)!}\right]$$

$$= \frac{2|\tau|(1-\beta)}{\gamma}e^{-m}\left[(k+1)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]$$

$$= \frac{2|\tau|(1-\beta)}{\gamma}e^{-m}[(k+1)me^{m}+(1-\alpha)(e^{m}-1)]$$

$$= \frac{2|\tau|(1-\beta)}{\gamma}[(k+1)m+(1-\alpha)(1-e^{-m})]$$

$\leq 1-\alpha$

by the given hypothesis.

This completes the proof of Theorem 2.1. □

Theorem 2.2. Let $f \in A$ be defined as in (1.1) suppose that $m > 0$, $k \geq 0$, $0 \leq \alpha < 1$ and the inequality

$$(k+1)\ \ 1-e^{-m}) - \frac{(k+\alpha)}{m}\ \ 1-e^{-m}-me^{-m}) \leq \frac{\gamma(1-\alpha)}{2|\tau|(1-\beta)}$$

is satisfied then for $f \in P_{\gamma}^{\tau}(\beta), 0 < \gamma \leq 1$ and $0 \leq \beta < 1$, $I(m)f(z) \in k - S_p(\alpha)$.

Proof. The proof of this theorem is much akin to that of Theorem 2.1 so we omit the details involved. □

Theorem 2.3. Let $m > 0$ be such that

$$\frac{2|\tau|(1-\beta)}{\gamma}\left[\ \ 1-e^{-m}) + \frac{(\lambda-1)}{m}\ \ 1-e^{-m}-me^{-m})\right] \leq \lambda$$

is satisfied then for $f \in P_{\gamma}^{\tau}(\beta);\ \ 0 < \gamma \leq 1,\ \ \beta < 1$ and $\lambda > 0$, $I(m)f(z) \in S_{\lambda}^{*}$.

Proof. To prove that $I(m)f(z) \in S_{\lambda}^{*}$, from Lemma 2.4 it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n+\lambda-1)|A_n| \leq \lambda$$

where

$$A_n = \frac{m^{n-1}}{(n-1)!}e^{-m}a_n, \quad n \geq 2.$$

Since $f \in P_{\gamma}^{\tau}(\beta)$ using Lemma 2.6 and $1 + \gamma(n-1) \geq \gamma n$ we need only to show that

$$\sum_{n=2}^{\infty}(n+\lambda-1)\frac{m^{n-1}}{(n-1)!}e^{-m}\frac{2|\tau|(1-\beta)}{1+\gamma(n-1)} \leq \lambda.$$

Now

$$\sum_{n=2}^{\infty}(n+\lambda-1)\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|$$
$$\le \sum_{n=2}^{\infty}(n+\lambda-1)\frac{m^{n-1}}{(n-1)!}e^{-m}\frac{2|\tau|(1-\beta)}{1+\gamma(n-1)}$$
$$\le \frac{2|\tau|(1-\beta)}{\gamma}\sum_{n=2}^{\infty}(n+\lambda-1)\frac{m^{n-1}}{n!}e^{-m}$$
$$= \frac{2|\tau|(1-\beta)}{\gamma}e^{-m}\left[\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}+\frac{(\lambda-1)}{m}\sum_{n=2}^{\infty}\frac{m^n}{n!}\right]$$
$$= \frac{2|\tau|(1-\beta)}{\gamma}\left[(1-e^{-m})+\frac{(\lambda-1)}{m}(1-e^{-m}-me^{-m})\right]$$
$$\le \lambda$$

by the given hypothesis. □

Thus the proof of Theorem 2.3 is complete.

Theorem 2.4. Let $m > 0$ and the inequality

$$\frac{2|\tau|(1-\beta)}{\gamma}\left[m+\lambda\ \ 1-e^{-m})\right] \le \lambda$$

is satisfied then for $f \in P_{\gamma}^{\tau}(\beta);\ \ 0<\gamma\le 1,\ \ \beta < 1$ and $\lambda > 0$, $I(m)f(z) \in C_{\lambda}$.

Proof. The proof is similar to that of Theorem 2.3 therefore we omit the details. □

Theorem 2.5. Let $m > 0$. If for $k \ge 0$, $0 \le \alpha < 1$ and the inequality

$$e^m\left[(1+k)m^3+(6+5k-\alpha)m^2+(7+4k-3\alpha)m\right] \le 1-\alpha$$

is satisfied then $I(m)f(z)$ maps $f(z) \in S$ of the form (1.1) into $k-UCV(\alpha)$.

Proof. Let $f(z) \in S$ be of the form (1.1). In view of Lemma 2.1 it is enough to show that

$$T=\sum_{n=2}^{\infty}n[n(k+1)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n| \le 1-\alpha.$$

Now

$$T=\sum_{n=2}^{\infty}n[n(k+1)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|$$
$$\le \sum_{n=2}^{\infty}n^2[n(k+1)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}$$
$$= \sum_{n=2}^{\infty}[(k+1)(n-1)(n-2)(n-3)+(5k+6-\alpha)(n-1)(n-2)$$
$$+(4k+7-3\alpha)(n-1)+(1-\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}$$
$$= (1+k)m^3+(6+5k-\alpha)m^2+(7+4k-3\alpha)m+(1-\alpha)(1-e^{-m})$$
$$\le 1-\alpha$$

by the given hypothesis.

Thus the proof of Theorem 2.5 is established. □

Theorem 2.6. Let $m > 0$ if for $k \ge 0$, $0 \le \alpha < 1$ and the inequality

$$e^m\left[(1+k)m^2+(3+2k-\alpha)m\right] \le 1-\alpha. \quad (2.7)$$

then $I(m)f(z)$ maps $f(z) \in S$ of the form (1.1) into $k-S_p(\alpha)$.

Proof. The proof of this theorem is much akin to that of Theorem 2.5. Therefore we omit the details involved. □

Theorem 2.7. Let $m > 0$, $\lambda > 0$ and the inequality

$$e^m\left[m^3+(\lambda+5)m^2+(3\lambda+4)m\right]\le\lambda$$

is satisfied then $I(m)f(z)$ maps $f(z) \in S$ of the form (1.1) into C_λ.

Proof. The proof of this theorem is similar to that of Theorem 2.3. Therefore we omit the details involved. □

III. AN INTEGRAL OPERATOR

In the following theorem, we obtain analogues results in connection with a particular integral operator $G(m,z)$ which is defined as follows

$$G(m,z)=\int_0^z\frac{I(m)f(t)}{t}dt. \tag{3.1}$$

Theorem 3.1. Let f be defined by (1.1) is in the class $P_\gamma^\tau(\beta)$ with $m > 0$ and the inequality

$$(k+1)(1-e^{-m})-\frac{k+\alpha}{m}(1-e^{-m}-me^{-m})\le\frac{\gamma(1-\alpha)}{2|\tau|(1-\beta)}$$

is satisfied then $G(m,z)$ defined by (3.1) is in the class $k-UCV(\alpha)$.

Proof. Since

$$G(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{n!}e^{-m}a_nz^n. \tag{3.2}$$

To prove it in $k-UCV(\alpha)$, we have to show that

$$\begin{aligned}T&=\sum_{n=2}^{\infty}n[n(k+1)-(k+\alpha)]|A_n|\\&\le\sum_{n=2}^{\infty}n[n(k+1)-(k+\alpha)]\frac{m^{n-1}}{n!}e^{-m}\left|\frac{2|\tau|(1-\beta)}{1+\gamma(n-1)}\right|\\&\le\frac{2|\tau|(1-\beta)}{\gamma}e^{-m}\sum_{n=2}^{\infty}[n(k+1)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}\\&=\frac{2|\tau|(1-\beta)}{\gamma}e^{-m}\sum_{n=2}^{\infty}[(k+1)(n-1)+(1-\alpha)]\frac{m^{n-1}}{(n-1)!}\\&=\frac{2|\tau|(1-\beta)}{\gamma}\left[(1+k)(1-e^{-m})+(1-\alpha)(1-e^{-m}-me^{-m})\right]\end{aligned}$$

$\le 1-\alpha$

by the given hypothesis.

This completes the proof of Theorem 3.1. □

Theorem 3.2. Let f be defined by (1.1) in the class S with $m > 0$ and the inequality (2.7) is satisfied then $G(m,z)$ defined by (3.1) is in $k-UCV(\alpha)$.

Proof. Since

$$G(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{n!}e^{-m}a_nz^n. \tag{3.3}$$

To prove that $G(m,z) \in k-UCV(\alpha)$, we have to show that

$$\sum_{n=2}^{\infty}n[n(k+1)-(k+\alpha)]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|\le 1-\alpha.$$

Using the well-known inequality $|a_n| \le n$ and proceeding the previous theorem we obtain the required condition.

□

The proof of following Theorems 3.3-3.5 are similar to Theorem 3.1 therefore we only state these theorems.

Theorem 3.3. Let *f* be defined by (1.1) in the class *S* with $m > 0$ and the inequality

$$(k+1)me^{m} \le 1-\alpha$$

is satisfied then $G(m,z)$ is in the class $k-S_p(\alpha)$.

Theorem 3.4. Let *f* be defined by (1.1) in the class *S* with $m > 0$ and the inequality

$$me^{m} \le \lambda$$

is satisfied then $G(m,z)$ is in the class $S^*(\lambda)$.

Theorem 3.5. Let *f* be defined by (1.1) in the class *S* with $m > 0$ and the inequality

$$e^m\left[m^2+(\lambda+2)m\right] \le \lambda$$

is satisfied then $G(m,z)$ is in the class $C(\lambda)$.

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