PROPERTIES OF SOME SUBCLASSES OF MULTIVALENT FUNCTIONS ASSOCIATED WITH CLOSE-TO-CONVEX FUNCTION

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ABSTRACT

For analytic function f(z) normalized with f(0)=0 and f'(0)=1 in the open unit disk U. A new class $L_1^*(\beta_1, \beta_2, \lambda)$ of f(z) satisfying some conditions with some complex number β_1, β_2 and some real number λ . The aim of present paper is to discuss some properties for $L_1^*(\beta_1, \beta_2, \lambda)$ of f(z) associated with close-to-convex in U.

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I INTRODUCTION AND DEFINITIONS

Let A denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p} \quad (a_n \in \mathbb{C}) \tag{1.1}$$

Which are analytic and p-valent in the open disc U = $\{z \in \mathbb{C} / |z| < 1\}$.

Let R(α) denote the subclass of A consisting of functions f(z) which satisfy

Re f'(z) $> \alpha$ ($z \in U$) for some real α $(0 \leq \alpha < 1)$.

A function f(z) $\in$ R(α) is said to be close-to-convex of order α in U (cf. Goodman[2])

We know that $R(\alpha_2) \subset R(\alpha_1)$ for $(0 \leq \alpha_1 \leq \alpha_2 < 1)$ and R(α) $\subset$ A by

Noshiro-Warshawski theorem (cf. Duren [3]).

Let $L_1^*(\beta_1, \beta_2, \lambda)$ denote the subclass of A defined as follow:

$$L_1^*(\beta_1, \beta_2, \lambda) = \left\{ f \in A : \left| \frac{f'(z) - p\, z^{p-1}}{\beta_1 f'(z) + \beta_2 p\, z^{p-1}} \right| \leq \lambda \right\}$$

For some complex β_1, β_2 and for some real λ.

Let T denote the subclass of A consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_n \geq 0) \tag{1.2}$$

Further, let $L^*(\beta_1, \beta_2, \lambda)$ denote the subclass of $L_1^*(\beta_1, \beta_2, \lambda)$ by

$$L^*(\beta_1, \beta_2, \lambda) = L_1^*(\beta_1, \beta_2, \lambda) \cap T$$

for some real number $\beta_1 (0 \leq \beta_1 \leq 1)$ *and* $\beta_2 (0 < \beta_2 \leq 1)$ and

for some real number $\lambda (0 < \lambda \leq 1)$.

The class $L^*(\beta_1, \beta_2, \lambda)$ was studied by Kim and Lee [4] for univalent function.

We note that :

1) $L^*\left(\beta_1, 1, \frac{1-\beta_1}{1+\beta_1{}^2}\right) = P^*(\beta_1)$, where $P^*(\beta_1)$ is the class of functions f(z)∈ T which satisfy Re
f'(z) > β_1. The class $P^*(\beta_1)$ was studied by Kim and Lee, Sarangi and Uralegaddi and Al. Amiri for univalent functions.

2) $L^*(0,1,\lambda) = G^*(\lambda)$,where $G^*(\lambda)$ is the class of functions f(z) ∈ T which satisfy $|f'(z) - 1| \leq \lambda$. The class $G^*(\lambda)$ was introduced by Kim and Lee for univalent function.

3) $L^*(1,1,\lambda) = D^*(\lambda)$, where $D^*(\lambda)$ is the class of functions $f(z) \epsilon T$ which satisfy

$\left|\frac{f'(z)-1}{f'(z)+1}\right| \leq \lambda$. The class $D^*(\lambda)$ was introduced by Kim and Lee [4] for univalent function.

II PROPERTIES OF THE CLASS $L_1^*(\beta_1, \beta_2, \lambda)$

1. Coefficient Estimate

First result for the class is contained in

Theorem 2.1

A function f(z) defined by equation 1.2 is in the class $L_1^*(\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} (1 + \lambda|\beta_1|) \; |p+n| \; |a_{n+p}| \leq \lambda \, p \, |(\beta_1 + \beta_2)| \tag{1.3}$$

Proof: It follows that

$$\left|\frac{f'(z) - p\, z^{p-1}}{\beta_1 f'(z) + \beta_2 p\, z^{p-1}}\right| = \left|\frac{-\sum_{n=1}^{\infty} (n+p)\, a_{n+p}\, z^n}{(\beta_1+\beta_2)p - \beta_1 \sum_{n=1}^{\infty} (n+p)\, z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} |n+p|\; |a_{n+p}|\; |z^n|}{|(\beta_1+\beta_2)p| - |\beta_1| \sum_{n=1}^{\infty} |n+p|\; |z^n|}$$

$$\leq \frac{\sum_{n=1}^{\infty} |n+p| \; |a_{n+p}|}{|(\beta_1+\beta_2)p| \; - \; |\beta_1| \; \sum_{n=1}^{\infty} |n+p|}$$

Therefore if f(z) satisfies the inequality (2.1) , then f(z)∈ $L_1^*(\beta_1, \beta_2, \lambda)$.

Conversely, it is simple to verify that if $f(z) \in L_1^*(\beta_1, \beta_2, \lambda)$, then

$$\sum_{n=1}^{\infty} (1 + \lambda|\beta_1|) \; |p+n| \; |a_{n+p}| \leq \lambda \, p \, |\beta_1 + \beta_2|$$

Corollary 2.1 If $f(z) \in L_1^*(\beta_1, \beta_2, \lambda)$ then we have

$$|a_{n+p}| \leq \frac{\lambda \, p \, |\beta_1+\beta_2|}{(1+\lambda|\beta_1|)\,(n+p)} (n = 1,2,3, \dots)$$

Corollary 2.2 A function $f(z)$ defined by (1.2) is in the class $L(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=1}^{\infty} (1 + \lambda\beta_1) \; (n+p) \; a_{n+p} \leq \lambda \, p \, (\beta_1 + \beta_2)$$

2. Distortion and covering theorem

Theorem 3.1 : If the function $f(z) \in L_1^*(\beta_1, \beta_2, \lambda)$ then

$$z^p - \sum_{n=1}^{\infty} \frac{\lambda p(\beta_1+\beta_2)}{(1+\lambda\beta_1)\,(n+p)} \; z^{n+p} \leq |f(z)| \leq z^p + \sum_{n=1}^{\infty} \frac{\lambda p(\beta_1+\beta_2)}{(1+\lambda\beta_1)\,(n+p)} \; z^{n+p} \qquad (3.1)$$

Proof: Now $f(z) \in L_1^*(\beta_1, \beta_2, \lambda)$ then

$$|f(z)| = |\, z^p - \sum_{n=1}^{\infty} a_{n+p} \, z^{n+p}|, \qquad (a_n \in C)$$

$$\leq |\, z^p| + \sum_{n=1}^{\infty} a_{n+p} \, |z^{n+p}|$$

But $|a_{n+p}| \leq \frac{\lambda \, p \, |\beta_1+\beta_2|}{(1+\lambda|\beta_1|)\,(n+p)} (n = 1,2,3, \dots)$

$|f(z)| \leq |\, z^p| + \sum_{n=1}^{\infty} \frac{\lambda p(\beta_1+\beta_2)}{(1+\lambda\beta_1)} |z^{n+p}|$ and also $|f(z)| \geq |\, z^p| - \sum_{n=1}^{\infty} \frac{\lambda p(\beta_1+\beta_2)}{(1+\lambda\beta_1)} |z^{n+p}|$

And hence $z^p - \sum_{n=1}^{\infty} \frac{\lambda p(\beta_1+\beta_2)}{(1+\lambda\beta_1)\,(n+p)} \; z^{n+p} \leq |f(z)| \leq z^p + \sum_{n=1}^{\infty} \frac{\lambda p(\beta_1+\beta_2)}{(1+\lambda\beta_1)\,(n+p)} \; z^{n+p}$

3. Modified Hadamard Product

Theorem 4.1 : If the functions $f(z)$ and $g(z) \in L^*(\beta_1, \beta_2, \lambda)$ then $f * g \in L^*(\beta_1, \beta_2, \lambda)$

For $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_n \geq 0)$ and $g(z) = z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \quad (b_n \geq 0)$

Then $f(z) * g(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} \; b_{n+p} z^{n+p}$

Where $\gamma > \frac{\lambda^2 p(\beta_1+\beta_2)}{(n+p)(1+\lambda\beta_1)^2 - \lambda^2 p\, \beta_1\, (\beta_1+\beta_2)}$

Proof: As $f(z)$ and $g(z) \in L^*(\beta_1, \beta_2, \lambda)$

$$\frac{(1+\lambda\beta_1)\ (n+p)}{\lambda\, p\, (\beta_1+\beta_2)}\, a_{n+p} \leq 1 \qquad \text{and} \qquad \frac{(1+\lambda\beta_1)\ (n+p)}{\lambda\, p\, (\beta_1+\beta_2)}\, b_{n+p} \leq 1$$

We have to find smallest number γ such that

$$\frac{(1+\gamma\beta_1)\ (n+p)}{\gamma\, p\, (\beta_1+\beta_2)}\, a_{n+p}\, b_{n+p} \leq 1$$

By Cauchy's –Schwarz inequality,

$$\frac{(1+\lambda\beta_1)\ (p+n)}{\lambda\, p\, (\beta_1+\beta_2)} \sqrt{a_{n+p} b_{n+p}} \leq 1 \qquad (4.1)$$

Therefore it is enough to show that

$$\frac{(1+\gamma\beta_1)\ (p+n)}{\gamma\, p\, (\beta_1+\beta_2)}\, a_{n+p}\, b_{n+p} \leq \frac{(1+\lambda\beta_1)\ (p+n)}{\lambda\, p\, (\beta_1+\beta_2)} \sqrt{a_{n+p} b_{n+p}}$$

That is $\sqrt{a_{n+p} b_{n+p}} \leq \frac{\gamma(1+\lambda\beta_1)}{\lambda(1+\gamma\beta_1)}$

From Equation (4.1)

$$\sqrt{a_{n+p} b_{n+p}} \leq \frac{\lambda\, p\, (\beta_1+\beta_2)}{(1+\lambda\beta_1)\ (n+p)}$$

Thus it is enough to show that

$$\frac{\lambda\, p\, (\beta_1+\beta_2)}{(1+\lambda\beta_1)\ (n+p)} \leq \frac{\gamma(1+\lambda\beta_1)}{\lambda(1+\gamma\beta_1)}$$

Which is simplifies to

$$\gamma > \frac{\lambda^2 p(\beta_1+\beta_2)}{(n+p)(1+\lambda\beta_1)^2 - \lambda^2 p\, \beta_1\, (\beta_1+\beta_2)}$$

4. Closure Theorem

Theorem 5.1 If $f_j(z) \in L^*(\beta_1, \beta_2, \lambda)$ $j = 1,2 \ldots\ldots s$ then $g(z) = \sum_{j=1}^{s} C_j f_j(z) \in L^*(\beta_1, \beta_2, \lambda)$

Where $f_j(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ and $\sum_{j=1}^{s} C_j = 1$

Proof: $g(z) = \sum_{j=1}^{s} C_j f_j(z)$

$$= z^p - \sum_{n=1}^{\infty} \sum_{j=1}^{s} C_j a_{n+p,j} z^{n+p}$$

$$= z^p - \sum_{n=1}^{\infty} e_k z^{n+p} \qquad \text{where} \quad e_k = \sum_{j=1}^{s} C_j a_{n+p,j}$$

Thus $g(z) \in L^*(\beta_1, \beta_2, \lambda)$ if $\sum_{n=1}^{\infty} \frac{(1+\lambda\beta_1)(n+p)}{\lambda p (\beta_1+\beta_2)} e_k \leq 1$

That is if $\sum_{n=1}^{\infty} \sum_{j=1}^{s} C_j a_{n+p,j} \frac{(1+\lambda\beta_1)(n+p)}{\lambda p (\beta_1+\beta_2)} \leq 1$

$$\sum_{j=1}^{s} C_j \sum_{n=1}^{\infty} \frac{(1+\lambda\beta_1)(n+p)}{\lambda p (\beta_1+\beta_2)} a_{n+p,j} \leq 1$$

$\Rightarrow f_j(z) \in L^*(\beta_1, \beta_2, \lambda)$

Theorem 5.2 : If $f(z), g(z) \in L^*(\beta_1, \beta_2, \lambda)$ then

$$h(z) = z^p - \sum_{n=1}^{\infty} \left[a_{n+p}^2 + b_{n+p}^2\right] z^{n+p} \in L^*(\beta_1, \beta_2, \lambda)$$

Where $\gamma \geq \frac{2\lambda^2 p (\beta_1+\beta_2)}{(1+\lambda\beta_1)^2 (n+p) - 2\lambda^2 \beta_1 p (\beta_1+\beta_2)}$

Proof: $f(z), g(z) \in L^*(\beta_1, \beta_2, \lambda)$ and so

$$\sum_{n=1}^{\infty} \left[\frac{(1+\lambda\beta_1)(n+p)}{\lambda p (\beta_1+\beta_2)}\right]^2 a_{n+p}^2 \leq \sum_{n=1}^{\infty} \left[\frac{(1+\lambda\beta_1)(n+p)}{\lambda p (\beta_1+\beta_2)}\right]^2 \leq 1 \qquad (5.1)$$

Similarly $\sum_{n=1}^{\infty} \left[\frac{(1+\lambda\beta_1)(n+p)}{\lambda p (\beta_1+\beta_2)}\right]^2 b_{n+p}^2 \leq 1$ (5.2)

We show that $\sum_{n=1}^{\infty} \left[\frac{(1+\gamma\beta_1)(n+p)}{\gamma p (\beta_1+\beta_2)}\right]^2 \left[a_{n+p}^2 + b_{n+p}^2\right]^2 \leq 1$

$\Rightarrow h(z) \in L^*(\beta_1, \beta_2, \lambda)$

Adding equation (5.1) and (5.2) , we get

$$\frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{(1+\gamma\beta_1)(n+p)}{\gamma p (\beta_1+\beta_2)}\right]^2 \left[a_{n+p}^2 + b_{n+p}^2\right]^2 \leq 1$$

That is enough to show

$$\left[\frac{(1+\gamma\beta_1)\ (n+p)}{\gamma\, p\,(\beta_1+\beta_2)}\right]^2 \le \frac{1}{2}\left[\frac{(1+\lambda\beta_1)\ (n+p)}{\lambda\, p\,(\beta_1+\beta_2)}\right]^2$$

By simplifying we get,

$$\gamma \ge \frac{2\,\lambda^2\,\mathrm{p}\,(\beta_1+\beta_2)}{(1+\lambda\beta_1)^2\ (n+p) - 2\,\lambda^2\,\beta_1 \mathrm{p}\,(\beta_1+\beta_2)}$$

5. Radius problem for the class R(α)

In this section, we discuss some radius problems for the class R(α). To discuss our problems, we need the following lemma for the class R(α).

Lemma 6.1 If $f(z)\ \epsilon\ R(\alpha)$ then, $\sum_{n=1}^{\infty}(n+p)|a_{n+p}| \le 1-\alpha$.

Corollary 6.1 If $f(z)\ \epsilon\ R(\alpha)$ then $|a_{n+p}| \le \frac{1-\alpha}{(n+p)} \le 1$

Remark 6.1 By lemma 6.1 we see that if $f(z)\ \epsilon\ R(\alpha)$ then

$$\sum_{n=1}^{\infty}(n+p)|a_{n+p}|^2 \le \sum_{n=1}^{\infty}(n+p)|a_{n+p}| \le 1-\alpha$$

Theorem 3.1 If $f(z)\ \epsilon\ R(\alpha)$ and $\delta \epsilon C$ $(0<|\delta|<1)$, then the function $\frac{1}{\delta}f(\delta z)$ belongs to the class $L_1^*(\beta_1,\beta_2,\lambda)$ for $(0<|\delta| \le |\delta_0(\lambda)|)$

Where $|\delta_0(\lambda)|$ is the smallest positive root of the equation

$$\mathrm{h}(|\delta|) = (1+\lambda\,|\beta_1|)|\delta|\sqrt{(1-\alpha)\ (2-|\delta|^2)} - \lambda\,\mathrm{p}\ |(\beta_1+\beta_2)|\ \ (1-|\delta|^2)$$

in $0<|\delta|<1$

Proof: For $f(z)\ \epsilon\ R(\alpha)$, we see that

$\frac{1}{\delta}f(\delta z) = z^p + \sum_{n=1}^{\infty}\delta^{n+p-1}\,z^{n+p}$ And $\sum_{n=1}^{\infty}(n+p)\ |a_{n+p}|^2 \le 1-\alpha$

To show that $f(z)\ \epsilon\ L_1^*(\alpha,\beta,\lambda)$, we need to prove that

$$\sum_{n=1}^{\infty}(1+\lambda|\beta_1|)\ |n+p|\ \ |a_{n+p}|\delta^{n+p-1} \le \lambda\,\mathrm{p}\,|(\beta_1+\beta_2)|$$

from theorem 2.1 Applying Cauchy-Schwarz inequality, we note that

$$\sum_{n=1}^{\infty}(1+\lambda|\beta_1|)\ (1+\lambda\beta_1)\,|n+p|\ \ |a_{n+p}|\ \ \delta^{n+p-1}$$

$$\leq \frac{[1+\lambda\beta_1]}{|\delta|}\left\{\left(\sum_{n=1}^{\infty}(n+p)\ |a_{n+p}|^2\right)^{1/2}\ \left(\sum_{n=1}^{\infty}\ (n+p)|\delta|^{2(n+p)}\right)^{1/2}\right\}$$

$$\leq \frac{[1+\lambda\beta_1]}{|\delta|}\ \left(\sum_{n=1}^{\infty}\ (n+p)\ |\delta|^{2(n+p)}\right)^{1/2}\ \sqrt{1-\alpha}$$

We note that

$$\sum_{n=1}^{\infty}\ (n+p)|\delta|^{2(n+p)} = \frac{|\delta|^4(2-|\delta|^2)}{(1-|\delta|^2)^2}$$

Therefore we show that

$$\sum_{n=1}^{\infty}(1+\lambda|\beta_1|)\ |n+p|\ \ |a_{n+p}|\ \ \delta^{n+p-1} \leq \frac{(1+\lambda\beta_1)|\delta|}{(1-|\delta|^2)}\sqrt{(1-\alpha)\ (2-|\delta|^2)}$$

Now, let us consider the complex $\delta \epsilon C$ $(0<|\delta|<1)$, such that

$$\frac{(1+\lambda|\beta_1|)|\delta|}{(1-|\delta|^2)}\sqrt{(1-\alpha)\ (2-|\delta|^2)} = \lambda\ \mathrm{p}|(\beta_1+\beta_2)|$$

If we define the function h($|\delta|$) by

$$\mathrm{h}(|\delta|) = (1\ +\ \lambda\ |\beta_1|)|\delta|\sqrt{(1-\alpha)\ (2-|\delta|^2)} - \lambda\mathrm{p}\ |(\beta_1+\beta_2)|\ \ (1-|\delta|^2)$$

then we have that

$$h(0) = -\lambda\ \mathrm{p}|(\beta_1+\beta_2)| < 0 \quad \text{and} \quad h(1) = (1\ +\ \lambda\ |\beta_1|)\sqrt{1-\alpha} > 0$$

This means that there exists some δ_0 such that $h(|\delta_0|) = 0$ $(0<|\delta_0|<1)$.

This completes the proof of the theorem.

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