FIXED POINTS OF COMPATIBILITY OF TYPE (β) IN MENGER SPACE

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ABSTRACT

The present paper deals with a common fixed point theorem for six self maps which generalizes the result of Pant and Chauhan [9], using the concept of compatibility of type (β) in Menger space.

Keywords and Phrases:* *Menger Space, Common Fixed Points, Compatible Maps, Compatible Maps of Type (β).

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I. INTRODUCTION

Jungck and Rhoades [6] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [13] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [5] soon enlarged this concept to compatible maps. Menger [7] introduced the notion of probabilistic metric space which is a generalization of metric space. It is also of fundamental importance in probabilistic functional analysis. The development of fixed point theory in PM-spaces was due to Schweizer and Sklar [11]. Sehgal and Bharucha-Reid [12] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

The notion of compatible mapping in a Menger space has been introduced by Mishra [8]. Cho, Murthy and Stojakovik [1] proposed the concept of compatible maps of type (A) in Menger space and gave some fixed point theorems. Recently, using the concept of compatible mappings of type (A), semi-compatibility and occasionally weak compatibility in Menger space, Jain et. al. [2, 3, 4] proved some interesting fixed point theorems in Menger space. In the sequel, Patel and Patel [10] proved a common fixed point theorem for four compatible maps of type (A) in Menger space by taking a new inequality.

In this paper a fixed point theorem for six self maps has been proved using the concept of mappings of compatibility of type (β). We also gave an example.

II. PRELIMINARIES

Definition 2.1.[7] A mapping $\mathcal{F} : \mathrm{R} \to \mathrm{R}^+$ is called a *distribution* if it is non-decreasing left continuous with

$\inf\{\mathcal{F}(t) \mid t \in R\} = 0$ and $\sup\{\mathcal{F}(t) \mid t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [2] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions :

(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a$, $d \ge b$,

(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [2] A *probabilistic metric space (PM-space)* is an ordered pair $(X, \mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,

for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [2] A *Menger space* is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \ge t\{F_{u,v}(x), F_{v,w}(y)\}$, for all $u, v, w \in X$, $x, y \ge 0$.

Definition 2.5. [11] A sequence $\{x_n\}$ in a Menger space $(X, \mathcal{F}, t)$ is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \text{ for all } n \ge M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x_m}(\varepsilon) > 1 - \lambda \quad \text{for all } m, n \ge M(\varepsilon, \lambda).$$

A Menger PM-space $(X, \mathcal{F}, t)$ is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way :

Proposition 2.1. [3] If (X, d) is a metric space then the metric d induces mappings $\mathcal{F} : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

$$H(k) = 0, \quad \text{for } k \le 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a,b) = \min\{a, b\}$. Then $(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.

The space $(X, \mathcal{F}, t)$ so obtained is called the *induced Menger space.*

Definition 2.6. [6] Self mappings A and S of a Menger space (X, $\mathcal{F}$, t) are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [8] Self mappings A and S of a Menger space (X, **F**, t) are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [1] Self maps S and T of a Menger space (X, $\mathcal{F}$, t) are said to be *compatible of type (β)* if $F_{SSx_n, TTx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.9. [9] Self maps S and T of a Menger space (X, $\mathcal{F}$, t) are said to be *semi-compatible* if $F_{STx_n, Tu}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

Now, the following example shows that the pair of self maps (I, L) are compatible of type (β) but not-semi-compatible.

Example 2.1. Let (X, d) be a metric space where X = [0, 2] and (X, $\mathcal{F}$, t) be the induced Menger space with $F_{x,y} = \dfrac{t}{t + d(x, y)}$ for all $t > 0$.

Define self maps I and L as follows :

$$I(x) = x \ \text{ for all } x \in X \ \text{ and } \ L(x) = \begin{cases} x, & \text{if } \ 0 \le x < 1 \\ 1, & \text{if } \ 1 \le x \le 2. \end{cases}$$

Taking $x_n = 1 - \dfrac{1}{n}$, we get $Ix_n = x_n = 1 - \dfrac{1}{n}$ and $Lx_n = 1 - \dfrac{1}{n}$.

Thus, $Lx_n \to 1$ as $n \to \infty$ and $Ix_n \to 1$, as $n \to \infty$.

Hence, $x = 1$

Since $Lx_n = 1 - \dfrac{1}{n}$

Therefore, $ILx_n = I\left(\dfrac{1}{2} - \dfrac{1}{n}\right) = 1 - \dfrac{1}{n}$

and $LLx_n = L\left(1 - \dfrac{1}{n}\right) = 1 - \dfrac{1}{n}$.

Also, $IIx_n = I\left(1 - \dfrac{1}{n}\right) = 1 - \dfrac{1}{n}$.

Consider $\lim\limits_{n\to\infty} F_{LLx_n, IIx_n}(t) = \lim\limits_{n\to\infty} F_{\frac{1}{2}-\frac{1}{n}, \frac{1}{2}-\frac{1}{n}}(t) = 1$ for $t > 0$.

Therefore, by definition, (I, L) is compatible mapping of type (β).

Now, $\lim\limits_{n\to\infty} F_{ILx_n, Lx}(t) = \lim\limits_{n\to\infty} F_{1-\frac{1}{n}, 1}(t) < 1$ for $t > 0$.

Therefore, (I, L) is not semi-compatible mapping. Thus the pair (I, L) of self maps is compatible of type (β) but not semi-compatible.

Remark 2.2. In view of above example, it follows that the concept of compatible maps of type (β) is more general than that of semi-compatible maps.

Lemma 2.1. [15] Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$ with continuous t-norms t and $t(a, a) \geq a$. If there exists a constant $k \in (0, 1)$ such that $F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t)$ for all $t \geq 0$ and $n = 1, 2, 3, \ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.3. [15] Let $(X, \mathcal{F}, t)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that

$F_{x, y}(kt) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

A class of implicit relation. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \rightarrow R$, non-decreasing in the first argument with the property :

a. For $u, v \geq 0$, $\phi(u, v, v, u) \geq 0$ or $\phi(u,v,u,v) \geq 0$ implies that $u \geq v$.

b. $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 2.3. Define $\phi(t_1,t_2,t_3,t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

III. MAIN RESULT

Theorem 3.1. Let A, B, L, M, S and T be self mappings on a complete Menger space $(X, \mathcal{F}, t)$ with $t(a, a) \geq a$, for some $a \in [0, 1]$, satisfying :

(3.1.1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;

(3.1.2) ST(X) and AB(X) are complete subspace of X;

(3.1.3) either AB or L is continuous;

(3.1.4) (L, AB) is compatible maps of type (β) and (M, ST) is weak compatible;

(3.1.5) for some $\phi \in \Phi$, there exists $k \in (0, 1)$ such that for $x, y \in X$ and $t > 0$,

$$\phi(F_{Lx, My}(kt), F_{ABx, STy}(t), F_{Lx, ABx}(t), F_{My, STy}(kt)) \geq 0$$

then A, B, L, M, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (3.1.1) $\exists$ $x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \quad \text{and} \quad Mx_1 = ABx_2 = y_1.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \quad \text{and} \quad Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$$

for $n = 0, 1, 2, \ldots$.

Step 1. Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mx_{2n+1}}(kt), F_{ABx_{2n}, STx_{2n+1}}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{y_{2n}, y_{2n+1}}(kt), F_{y_{2n-1}, y_{2n}}(t), F_{y_{2n}, y_{2n-1}}(t), F_{y_{2n+1}, y_{2n}}(kt)) \geq 0.$$

Using (a), we get

$$F_{y_{2n}, y_{2n+1}}(kt) \geq F_{y_{2n-1}, y_{2n}}(t).$$

Therefore, for all n even or odd, we have

$$F_{y_n, y_{n+1}}(kt) \geq F_{y_{n-1}, y_n}(t).$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X, which is complete.

Hence $\{y_n\} \rightarrow z \in X$. Also its subsequences converges as follows :

$\{Lx_{2n}\} \to z$, $\{ABx_{2n}\} \to z$, $\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$.

Case I. When AB is continuous.

As AB is continuous, $(AB)^2x_{2n} \to ABz$ and $(AB)Lx_{2n} \to ABz$.

As (L, AB) is compatible pair of type (β), so

$LLx_{2n} \to (AB)(AB)x_{2n}$ and so $LABx_{2n} \to ABz$

Step 2. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LABx_{2n}, Mx_{2n+1}}(kt), F_{ABABx_{2n}, STx_{2n+1}}(t), F_{LABx_{2n}, ABABx_{2n}}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(F_{ABz, z}(kt), F_{ABz, z}(t), F_{ABz, ABz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{ABz, z}(kt), F_{ABz, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{ABz, z}(kt), F_{ABz, z}(t), 1, 1) \geq 0.$$

Using (b), we get

$F_{ABz, z}(t) = 1$, for all $t > 0$,

i.e. $ABz = z$.

Step 3. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz, Mx_{2n+1}}(kt), F_{ABz, STx_{2n+1}}(t), F_{Lz, ABz}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Lz, z}(kt), F_{ABz, z}(t), F_{Lz, ABz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{Lz, z}(kt), 1, F_{Lz, z}(t), 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lz, z}(kt), 1, F_{Lz, z}(t), 1) \geq 0.$$

Using (a), we get

$F_{z, Lz}(kt) = 1$, for all $t > 0$,

i.e. $z = Lz$.

Thus, we have $z = Lz = ABz$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LBz, Mx_{2n+1}}(kt), F_{ABBz, STx_{2n+1}}(t), F_{LBz, ABBz}(t), F_{Mx_{2n+1}, STx_{2n}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Bz, z}(kt), F_{Bz, z}(t), F_{Bz, Bz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{Bz, z}(kt), F_{Bz, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Bz, z}(t), F_{Bz, z}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{Bz, z}(t) = 1$, for all $t > 0$,

i.e. $z = Bz$.

Since $z = ABz$, we also have

$z = Az$.

Therefore, $z = Az = Bz = Lz$.

Step 5. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$z = Lz = STv.$$

Putting $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mv}(kt), F_{ABx_{2n}, STv}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mv, STv}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{z, Mv}(kt), F_{z, STv}(t), F_{z, z}(t), F_{Mv, z}(kt)) \geq 0$$

$$\phi(F_{z, Mv}(kt), 1, 1, F_{z, Mv}(kt)) \geq 0$$

Using (a), we have

$$F_{z, Mv}(kt) \geq 1, \text{ for all } t > 0.$$

Hence, $F_{z, Mv}(t) = 1$.

Thus, $z = Mv$.

Therefore, $z = Mv = STv$.

As (M, ST) is weakly compatible, we have

$$STMv = MSTv. \qquad \text{Thus, } STz = Mz.$$

Step 6. Putting $x = x_{2n}$ and $y = z$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Mz}(t), F_{z, Mz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Mz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Mz}(t) = 1$, we have

$z = Mz = STz$.

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) and using Step 5, we get

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Lz, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(t), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{z,\ Tz}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\ Tz}(t) = 1$, we have

$z = Tz$.

Since $Tz = STz$, we also have $z = Sz$.

Hence

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Case II. When L is continuous.

As L is continuous, $L^2x_{2n} \rightarrow Lz$ and $L(AB)x_{2n} \rightarrow Lz$.

As (L, AB) is compatible map of type (β), so

$$LLx_{2n} \rightarrow (AB)\,(AB)x_{2n} \text{ and } LABx_{2n} \rightarrow ABz$$

By uniqueness of limit in Menger space, we have

$$Lz = ABz.$$

Step 8. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz,\ Mx_{2n+1}}(kt), F_{ABz,\ STx_{2n+1}}(t), F_{Lz,\ ABz}(t), F_{Mx_{2n+1},\ STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Lz,\ z}(kt), F_{Lz,\ z}(t), F_{Lz,\ Lz}(t), F_{z,\ z}(kt)) \geq 0$$

$$\phi(F_{Lz,\ z}(kt), F_{Lz,\ z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lz,\ z}(t), F_{Lz,\ z}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{z,\ Lz}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\ Lz}(t) = 1$

$\Rightarrow$ $z = Lz$.

Therefore,

$z = Lz = ABz$.

Step 9. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LBz,\ Mx_{2n+1}}(kt), F_{ABBz,\ STx_{2n+1}}(t), F_{LBz,\ ABBz}(t), F_{Mx_{2n+1},\ STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Bz,\ z}(kt), F_{Bz,\ z}(t), F_{Bz,\ Bz}(t), F_{z,\ z}(kt)) \geq 0$$

$$\phi(F_{Bz,\ z}(kt), F_{Bz,\ z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Bz,\ z}(t), F_{Bz,\ z}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{Bz,\ z}(t) \geq 1$, for all $t > 0$.

Thus, $F_{Bz,\ z}(t) = 1$

$\Rightarrow \quad z = Bz.$

Since $z = ABz$, we also have $z = Az$.

Therefore, $\quad z = Az = Bz = Lz.$

Step 10. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$z = Lz = STv.$$

Putting $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mv}(kt), F_{ABx_{2n}, STv}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mv, STv}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{z, Mv}(kt), F_{z, STv}(t), F_{z, z}(t), F_{Mv, z}(kt)) \geq 0$$

$$\phi(F_{z, Mv}(kt), 1, 1, F_{z, Mv}(kt)) \geq 0$$

Using (a), we have

$$F_{z, Mv}(kt) \geq 1, \text{ for all } t > 0.$$

Hence, $F_{z, Mv}(t) = 1$.

Thus, $z = Mv$.

Therefore, $z = Mv = STv$.

As (M, ST) is weakly compatible, we have

$$STMv = MSTv.$$

Thus, $STz = Mz$.

Step 11. Putting $x = x_{2n}$ and $y = z$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Mz}(t), F_{z, Mz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Mz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Mz}(t) = 1$, we have

$$z = Mz = STz.$$

Step 12. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) and using Step 5, we get

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Lz, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{z,\ Tz}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\ Tz}(t) = 1$, we have

$z = Tz$.

Since $Tz = STz$, we also have $z = Sz$.

Hence $Az = Bz = Lz = Mz = Tz = Sz = z$.

Hence, the six self maps have a common fixed point in this case also.

Uniqueness. Let w be another common fixed point of A, B, L, M, S and T; then $w = Aw = Bw = Lw = Mw = Sw = Tw$.

Putting $x = z$ and $y = w$ in (3.1.5), we get

$$\phi(F_{Lz,\ Mw}(kt), F_{ABz,\ STw}(t), F_{Lz,\ ABz}(t), F_{Mw,\ STw}(kt)) \geq 0$$

$$\phi(F_{z,\ w}(kt), F_{z,\ w}(t), F_{z,\ z}(t), F_{w,\ w}(kt)) \geq 0$$

$$\phi(F_{z,\ w}(kt), F_{z,\ w}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z,\ w}(t), F_{z,\ w}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{z,\ w}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\ w}(t) = 1$,

i.e., $z = w$.

Therefore, z is a unique common fixed point of A, B, L, M, S & T.

This completes the proof.

Remark 3.1. The above theorem is a generalization of the result of Pant et. al. [9] in the sense that the condition of semi-compatibility has been replaced by compatibility of type (β).

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