SOME RESULTS ON WEAK CO-MULTIPLICATION MODULES

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ABSTRACT

Let $R$ be a commutative ring with identity and all modules to be treated as unitary modules. In this paper we obtain some results on weak co-multiplication modules.

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I. INTRODUCTION

Multiplication module was introduced by Barnard [8] in 1981. The dual notion of multiplication module as co-multiplication module was introduced by Ansari- Toroghy and Farshadifar [6] in 2007. Using the concept prime sub-module of module, the concept of weak multiplication module was developed and many more results have been given by Azizi Shiraz [4]. In the year 2009 the dual notion of weak multiplication module as weak co-multiplication module was introduced by Atani and Atani [5]. Some results on co-multiplication module were given by Saeed Rajaee [1]. This paper continues this line of research for weak co-multiplication modules.

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unitary. If $N$ and $K$ are submodules of $R$-module $M$ then the residual ideal $N$ by $K$ is defined as $N : R K = \{ r \in R : r K \subseteq N \}$. Let $N$ be submodule of $M$ and $I$ be an ideal of $R$ the residual submodule $N$ by $I$ is defined as $N : M I = \{ m \in M : mI \subseteq N \}$.

In the special case in which $N = 0$ the ideal $(0 : R K)$ is called annihilator of $K$ and it is denoted by Ann$_R(K)$ also the submodule $(0 : M I)$ is called the annihilator of $I$ in $M$ and it is denoted by Ann$_M(I)$. A proper submodule $N$ of an $R$-module $M$ is said to be prime submodule of module $M$ if $ra \in N$ for $r \in R$ and $a \in M$ then either $a \in N$ or $rM \subseteq N$[10] (also see examples in [11], [12].) The set of all prime submodules in an $R$-module $M$ is denoted by Spec$(M)$.

The aim of this paper is to investigate some results on weak co-multiplication modules.

II. PRELIMINARIES

In this section we give some basic definitions which will be helpful to understand the further results.

Definition 2.1 [8] An $R$-module $M$ is said to be a multiplication module if for every submodule $N$ of $M$, there exist an ideal $I$ of $R$ such that $N = I M$.
Definition 2.2 [6] An $R$-module $M$ is said to be co-multiplication module if for every submodule $N$ of $M$ there exist an ideal $I$ of $R$ such that $N = (0 :_M I)$. It also follows that $M$ is a co-multiplication module if and only if $N = (0 :_M \text{Ann}_R(N))$ for every submodule $N$ of $M$.

Definition 2.3 [4] An $R$-module $M$ is called weak multiplication module if $M$ doesn’t have any prime submodule or every prime submodule $N$ of $M$, we have $N = I : M$, where $I$ is an ideal of $R$.

One can easily show that if an $R$-module $M$ is a weak multiplication module then $N = (N :_R M)$ for every prime submodule $N$ of $M$ [9].

Definition 2.4 [5] Let $R$ be a commutative ring. An $R$-module $M$ is defined to be a weak co-multiplication module if $\text{Spec}(M) = \emptyset$ or for every prime submodule $N$ of $M$, $N = (0 :_M I) = \text{Ann}_M(I)$ for some ideal $I$ of $R$. Also $M$ is a weak co-multiplication module if and only if $N = [0 :_M \text{Ann}_R(N)]$ for every prime submodule $N$ of $M$. We denote this concept by $N \subseteq WC M$.

Definition 2.5[2] A submodule $N$ of an $R$-module $M$ is said to be pure submodule if $IN = N \cap IM$, for every ideal $I$ of $R$.

Definition 2.6 [3] A submodule $N$ of an $R$-module $M$ is said to be co-pure submodule if $(N :_M I) = N + (0 :_M I)$, for every ideal $I$ of $R$.

Definition 2.7[1] An $R$-module $M$ is said to be fully pure (respectively fully co-pure) if every submodule of $M$ is pure (respectively co-pure).

Definition 2.8 [7] If $R$ is a ring and $M$ is an $R$-module then $M$ is said to be semisimple module if every submodule of $M$ is a direct summand of $M$.

III. MAIN RESULTS

In this section we obtain some results on weak co-multiplication modules.

Proposition 3.1

Let $M$ be an $R$-module and $N \subseteq L \subseteq M$ then $L/N$ is a weak co-multiplication submodule of $M/N$ if and only if there exists an ideal $I$ of $R$ such that

$L/N = [N :_M I] = \{ m + N \in M/N | (m + N) = \text{Im} + N \subseteq N \} = \{ m + N \in M/N | \text{Im} \subseteq N \} = \{ m + N \in M/N | m \in [N :_M I] \} = \text{Ann}_M(I) + N$.

Proof:

Since $IN \subseteq N$ for every ideal $I$ of $R$, hence $N \subseteq L = [N :_M I]$. We consider $M/N$ as an $R$-module. If $L/N$ is a weak co-multiplication module then $\text{Spec}(M/N) = \emptyset$ or for every prime submodule $L/N \subseteq WC M/N$ then there exists an ideal $I$ of $R$ such that

$L/N = [N :_M I] = \{ m + N \in M/N | (m + N) = \text{Im} + N \subseteq N \} = \{ m + N \in M/N | \text{Im} \subseteq N \} = \{ m + N \in M/N | m \in [N :_M I] \} = [N :_M I] / N$. Therefore $L = [N :_M I]$. The converse is clearly true.

Further let $N$ be co-pure then $[N :_M I] = N + [0 :_M I]$. So

$L/N = N + [0 :_M I] / N \cong [0 :_M I] / N \cap [0 :_M I] = [0 :_M I] / [0 :_N I] = \text{Ann}_M(I) / \text{Ann}_N(I)$.

In particular let $M$ be a semisimple $R$-module then there exists $K \subseteq M$ such that $M = N \oplus K$. Therefore $[N :_M I] = [K :_R I] + [N :_N I] = [0 :_K I] + N \subseteq [0 :_M I] + N$. Conversely it is clear that $[0 :_M I] + N \subseteq [N :_M I]$. Therefore $[N :_M I] = [0 :_M I] + N$ and hence $N$ is co-pure.
Corollary 3.2
Let \( M \) be an \( R \)-module and \( N \subseteq L \subseteq M \). If \( N \subseteq_{WC} M \) and \( M/N \) be weak co-multiplication \( R \)-module then \( L \subseteq_{WC} M \).

**Proof:**
We suppose that \( N = \{ 0 :_M J \} \) for some ideal \( J \) of \( R \). Since \( L/N \subseteq_{WC} M/N \) (by above Proposition 3.1), we have \( L = \{ N :_M I \} \) for some ideal \( I \) of \( R \). Therefore \( L = \{ 0 :_M IJ \} \) and hence \( L \subseteq_{WC} M \). This completes the proof.

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