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FIXED POINT THEOREM IN MENGER SPACE VIA OCCASIONALLY WEAK COMPATIBLE MAPPINGS

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ABSTRACT

The object of this paper is to establish a unique common fixed point theorem for six self mappings using the concept of occasionally weak-compatibility in Menger space which is an alternate result of Pant et. al. [8].

Keywords and Phrases. ***Menger space, Common fixed points, Compatible maps, and Occasionally Weak compatibility.***

AMS Subject Classification (2000). Primary 47H10, Secondary 54H25.

I. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [6]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [9] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [10] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [5] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [11] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [4] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [7]. The concept of weak compatible mappings is most general among all the commutativity concepts in this field as every pair of R-weakly commuting maps is compatible and each pair of compatible maps is weak-compatible but the reverse is not true always.

The intent of this paper is to generalize the result of Pant et. al. [8]. So, our generalization in this paper is two fold as

(i) Relaxed continuity of maps completely

(ii) Weakened the concept of semi-compatibility by a more general concept of occasionally weak compatible.

II. PRELIMINARIES

Definition 2.1.[7] A mapping $\mathcal{F} : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with

$$\inf \{ \mathcal{F}(t) \mid t \in R \} = 0 \qquad \text{and} \qquad \sup \{ \mathcal{F}(t) \mid t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [7] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions :

(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a$, $d \ge b$,

(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [7] A *probabilistic metric space (PM-space)* is an ordered pair $(X, \mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u, v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,

for all $u, v, w \in X$ and $x, y > 0$.

Definition 2.4. [7] A *Menger space* is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \ge t \{F_{u, v}(x), F_{v, w}(y) \}$, for all $u, v, w \in X$, $x, y \ge 0$.

Definition 2.5. [9] A sequence $\{x_n\}$ in a Menger space $(X, \mathcal{F}, t)$ is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \text{ for all } n \ge M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x_m}(\varepsilon) > 1- \lambda \qquad \text{for all } m, n \ge M(\varepsilon, \lambda).$$

A Menger PM-space $(X, \mathcal{F}, t)$ is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way :

Proposition 2.1. [7] If (X, d) is a metric space then the metric d induces mappings $\mathcal{F} : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

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$H(k) = 0, \quad \text{for } k \le 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a,b) = \min\{a, b\}$. Then $(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.

The space $(X, \mathcal{F}, t)$ so obtained is called the *induced Menger space.*

Definition 2.6. [2] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. Two self mappings f and g of a Menger space $(X, \mathcal{F}, t)$ are said to be occasionally weak compatible if there is a point $x \in X$ which is coincidence point of f and g at which f and g commute.

Definition 2.8. [3] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [8] Self maps S and T of a Menger space $(X, \mathcal{F}, t)$ are said to be *semi-compatible* if $F_{STx_n, Tu}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

It follows that if (S, T) is semi compatible and $Sx = Tx$ then $STx = TSx$. Thus if the pair (S, T) is semi-compatible then it is occasionally weakly compatible. The converse is not true.

Remark 2.1. Every semi-compatible pair of self-maps is occasionally weak compatible but the reverse is not true always.

Lemma 2.1. [13] Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$ with continuous t norms t and $t(a, a) \ge a$. If there exists a constant $k \in (0, 1)$ such that $F_{x_n, x_{n+1}}(kt) \ge F_{x_{n-1}, x_n}(t)$ for all $t \ge 0$ and $n = 1, 2, 3, \ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2. [13] Let $(X, \mathcal{F}, t)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that $F_{x, y}(kt) \ge F_{x, y}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Proposition 2.2. In a Menger space $(X, \mathcal{F}, t)$ if $t(x, x) \ge x$, $\forall\, x \in [0, 1]$ then $t(a, b) = \min\{a, b\}$, $\forall\, a, b \in [0, 1]$.

Proposition 2.3. Let $\{x_n\}$ be a Cauchy sequence in a Menger space $(X, \mathcal{F}, t)$ with continuous t-norm t. If the subsequence $\{x_{2n}\}$ converges to x in X, then $\{x_n\}$ also converges to x.

Proof. As $\{x_{2n}\}$ converges to x, we have

$$F_{x_n,x}(\varepsilon) \ge t\left(F_{x_n,x_{2n}}\left(\frac{\varepsilon}{2}\right), F_{x_{2n},x}\left(\frac{\varepsilon}{2}\right)\right).$$

Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} F_{x_n, x}(\varepsilon) \ge t(1, 1)$, which gives

$\lim_{n\to\infty} F_{x_n, x}(\varepsilon) = 1$; for all $\varepsilon > 0$ and the result follows.

A class of implicit relation. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \to R$, non-decreasing in the first argument with the property :

a. For $u, v \ge 0$, $\phi(u, v, v, u) \ge 0$ or $\phi(u,v,u,v) \ge 0$ implies that $u \ge v$.

b. $\phi(u, u, 1, 1) \ge 0$ implies that $u \ge 1$.

Example 2.1. Define $\phi(t_1,t_2,t_3,t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi\in\Phi$.

III. MAIN RESULT

Theorem 3.1. Suppose A, B, L, M, S and T be self mappings on a Menger space $(X, \mathcal{F}, t)$ with continuous t-norm t satisfying :

(3.1.1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;

(3.1.2) AB = BA, ST = TS, LB = BL, MT = TM;

(3.1.3) One of ST(X), M(X), AB(X) or L(X) is complete;

(3.1.4) The pairs (L, AB) and (M, ST) are occasionally weak-compatible;

(3.1.5) for some $\phi \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\phi(F_{Lx,\,My}(kt), F_{ABx,\,STy}(t), F_{Lx,\,ABx}(t), F_{My,\,STy}(kt)) \geq 0$$

then A, B, L, M, S and T have a unique common fixed point in X.

Proof. Suppose $x_0 \in X$. From condition (3.1.1) $\exists$ $x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \quad \text{and} \quad Mx_1 = ABx_2 = y_1.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \quad \text{and} \quad Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$$

for $n = 0, 1, 2, \ldots$.

Step 1. On putting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lx_{2n},\,Mx_{2n+1}}(kt), F_{ABx_{2n},\,STx_{2n+1}}(t), F_{Lx_{2n},\,ABx_{2n}}(t), F_{Mx_{2n+1},\,STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{y_{2n},\,y_{2n+1}}(kt), F_{y_{2n-1},\,y_{2n}}(t), F_{y_{2n},\,y_{2n-1}}(t), F_{y_{2n+1},\,y_{2n}}(kt)) \geq 0.$$

Using (a), we get

$$F_{y_{2n},\,y_{2n+1}}(kt) \geq F_{y_{2n-1},\,y_{2n}}(t).$$

Therefore, for all n even or odd, we have

$$F_{y_n,\,y_{n+1}}(kt) \geq F_{y_{n-1},\,y_n}(t).$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X.

Case I. ST(X) is complete.

In this case $\{y_{2n}\} = \{STx_{2n+1}\}$ is a Cauchy sequence in ST(X), which is complete. Thus $\{y_{2n+1}\}$ converges to some $z \in ST(X)$.

By Proposition 2.3, we have

$$\{Mx_{2n+1}\} \to z \quad \text{and} \quad \{STx_{2n+1}\} \to z$$

$$\{Lx_{2n}\} \to z \quad \text{and} \quad \{ABx_{2n}\} \to z.$$

As $z \in ST(X)$ there exists $u \in X$ such that $z = STu$:

Step I. Putting $x = x_{2n}$ and $y = u$ in (3.1.5) we get,

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$$\phi(F_{Lx_{2n}, Mu}(kt), F_{ABx_{2n}, STu}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mu, STu}(kt)) \geq 0.$$

Taking limit as $n \to \infty$, we get

$$\phi(F_{z, Mu}(kt), F_{z, z}(t), F_{z, z}(t), F_{Mu, z}(kt)) \geq 0$$

$$\phi(F_{z, Mu}(kt), 1, 1, F_{z, Mu}(kt)) \geq 0$$

Using (a) we have $F_{z, Mu}(kt) \geq 1$, for all $t > 0$.

Hence $F_{z, Mu}(t) = 1$.

Thus $z = Mu$.

Hence $STu = Mu = z$. As (M, ST) is occasionally weak-compatible so we have

$$Mz = STz.$$

Step II. Putting $x = x_{2n}$ and $y = z$ in (3.1.5) we get,

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Taking limit as $n \to \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, z}(t), F_{z, z}(t), F_{Mz, z}(kt)) \geq 0$$

$$\phi(F_{z, Mz}(kt), 1, 1, F_{z, Mz}(kt)) \geq 0$$

Using (a) we have

$$F_{z, Mz}(kt) \geq 1, \text{ for all } t > 0.$$

Hence $F_{z, Mz}(t) = 1$.

Thus $z = Mz$.

Step III. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) we get,

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

As $MT = TM$ and $ST = TS$ we have $MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$.

Letting $n \to \infty$, we get

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(t), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we get

$$F_{z, Tz}(t) \geq 1 \text{ for all } t > 0.$$

Hence,

$$F_{z, Tz}(t) = 1, \text{ for all } t > 0,$$

i.e. $z = Tz$.

Now $STz = Tz = z$ implies $Sz = z$. Hence $Sz = Tz = Mz = z$.

Step IV. As $M(X) \subseteq AB(X)$ there exists $v \in X$ such that $z = Mz = ABv$.

Putting $x = v$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lv,\, Mx_{2n+1}}(kt), F_{ABv,\, STx_{2n+1}}(t), F_{Lv,\, ABv}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

Letting n $\to \infty$, we get

$$\phi(F_{Lv,\, z}(kt), F_{z,\, z}(t), F_{Lv,\, z}(t), F_{z,\, z}(kt)) \geq 0$$

$$\phi(F_{Lv,\, z}(kt), 1, F_{Lv,\, z}(t), 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lv,\, z}(t), 1, F_{Lv,\, z}(t), 1) \geq 0.$$

Using (a), we have

$$F_{Lv,\, z}(t) \geq 1, \text{ for all } t > 0$$

which gives $Lv = z$.

Therefore, $ABz = Lz$.

Step V. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz,\, Mx_{2n+1}}(kt), F_{ABz,\, STx_{2n+1}}(t), F_{Lz,\, ABz}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

Letting n $\to \infty$, we get

$$\phi(F_{Lz,\, z}(kt), F_{Lz,\, z}(t), F_{Lz,\, Lz}(t), F_{z,\, z}(kt)) \geq 0$$

$$\phi(F_{Lz,\, z}(kt), F_{Lz,\, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lz,\, z}(t), F_{Lz,\, z}(t), 1, 1) \geq 0.$$

Using (b), we have $F_{Lz,\, z}(t) \geq 1$, for all $t > 0$

which gives $Lz = z$.

Therefore, $ABz = Lz = z$.

Step VI. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LBz,\, Mx_{2n+1}}(kt), F_{ABBz,\, STx_{2n+1}}(t), F_{LBz,\, ABBz}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

As $BL = LB$, $AB = BA$, so we have $L(Bz) = B(Lz) = Bz$ and $AB(Bz) = B(ABz) = Bz$. Letting n $\to \infty$, we get

$$\phi(F_{Bz,\, z}(kt), F_{Bz,\, z}(t), F_{Bz,\, Bz}(t), F_{z,\, z}(kt)) \geq 0$$

$$\phi(F_{Bz,\, z}(kt), F_{Bz,\, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Bz,\, z}(t), F_{Bz,\, z}(t), 1, 1) \geq 0.$$

Using (b), we have $F_{Bz,\, z}(t) \geq 1$, for all $t > 0$

which gives $Bz = z$ and $ABz = z$ implies $Az = z$.

Therefore $Az = Bz = Lz = z$.

Combining the results from different steps, we have

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence the six self maps have a common fixed point in this case. Case when $L(X)$ is complete follows from above case as $L(X) \subseteq ST(X)$.

Case II. AB(X) is complete. This case follows by symmetry. As $M(X) \subseteq AB(X)$, therefore the result also holds when M(X) is complete.

Uniqueness. Let u be another common fixed point of A, B, L, M, S and T, then

$$Au = Bu = Lu = Su = Tu = Mu = u.$$

Putting x = z and y = w in (3.1.5), we get

$$\phi(F_{Lz,\, Mw}(kt), F_{ABz,\, STw}(t), F_{Lz,\, ABz}(t), F_{Mw,\, STw}(kt)) \geq 0$$

$$\phi(F_{z,\, w}(kt), F_{z,\, w}(t), F_{z,\, z}(t), F_{w,\, w}(kt)) \geq 0$$

$$\phi(F_{z,\, w}(kt), F_{z,\, w}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z,\, w}(t), F_{z,\, w}(t), 1, 1) \geq 0.$$

Using (b), we have $F_{z,\, w}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\, w}(t) = 1$,

i.e., $z = w$.

Therefore, z is a unique common fixed point of A, B, L, M, S and T.

This completes the proof.

Remark 3.1. In view of proposition 2.2, t(a, b) = min{a, b}, theorem 3.1 is an alternate result of Pant et. al. [8], reducing the semi-compatibility of the pair (L, AB) to its occasionally weak compatibility and dropping the condition of continuity in a Menger space with continuous t-norm.

If we take B = T = I, the identity map in theorem 3.1, we get the following corollary.

Corollary 3.1. Let A, L, M and S be self mappings on a Menger space $(X, \mathcal{F}, t)$ with continuous t-norm t satisfying :

(3.1.6) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;

(3.1.7) One of S(X), M(X), A(X) or L(X) is complete;

(3.1.8) The pairs (L, A) and (M, S) are occasionally weak-compatible;

(3.1.9) for some $\phi \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\phi(F_{Lx,\, My}(kt), F_{Ax,\, Sy}(t), F_{Lx,\, Ax}(t), F_{My,\, Sy}(kt)) \geq 0$$

then A, L, M and S have a unique common fixed point in X.

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