SOME FIXED POINT THEOREMS ON PRODUCT SPACES BY THE SETTING OF RHOADES

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ABSTRACT

Rhoades discussed a number of fixed point theorems dealing with contractive conditions with rational expressions. In an analogous manner we define mappings on product spaces which satisfy such contractive like conditions in the first variable, and generalize the result of Nadler to such mappings. Here we discuss only those conditions which involve a single mapping. In the Nadler's result we enlarge the class of mappings by Rhoades type contractive conditions in the first variable and the class of metric spaces Z by the class of uniform spaces.

Keywords: Complete Metric Space, Fixed Point Property, Locally Compact Space, Product Space , Uniformly Continuous Mapping.

I. INTRODUCTION

The fixed point property (f. p. p.) is not necessarily preserved under the Cartesian product of spaces [2,3]. It is preserved when the maps f: XxZ $\rightarrow$XxZ have special contraction properties. Nadler's results are in this direction. Nadler's main results are as follows:

1.1 Theorem

Let (X, d) be a metric space. Let A_i: X$\rightarrow$X be a function with at least one fixed point a_i for each i=1, 2, ……….., and let A_0:X$\rightarrow$X be a contraction mapping with fixed point a_0. If the sequence {A_i} converges uniformly to A_0, then the sequence {a_i} converges to a_0.

1.2 Theorem

Let (X, d) be a locally compact metric space, let A_i: X$\rightarrow$X be a contraction mapping with fixed point a_i for each i=1,2,…….. If the sequence {A_i} converges pointwise to A_0, then the sequence {a_i} converges to a_0.

1.3 Theorem

Let (X, d) be a complete metric space, Z a metric space which has the f.p.p. and f: XxZ$\rightarrow$XxZ be a contraction in the first variable.

(a) If f is uniformly continuous, then f has a fixed point.

(b) If (X, d) is locally compact, f is continuous, then f has a fixed point.

In what follows, X will denote a complete metric space, Z a uniform space in which sequences are adequate and f a mapping of XxZ into XxZ. For a fixed z$\in$Z, f_z: X$\rightarrow$X be a mapping which is defined as $f_z(x)=\pi_1 f(x, z)$

for all $x \in X$, where π_1 is the projection of XxZ on X along Z. $(m)'$, $1 \leq m \leq 10$; will denote the condition $(m)'$ in Rhoades [7] with the modification that constant or functions that appear in $(m)'$ depend on z.

II. SOME DEFINITIONS FROM RHOADES [7]

Let (X, d) be a complete metric space and $f: X \to X$ be mapping. For $x \in X$, let $0(x)=\{x, f(x), f^2(x), \ldots\ldots\}$ be the orbit of x under f. Consider the following conditions on f and (X, d):

(1)' (Dass and Gupta) – There exist numbers $\alpha, \beta > 0$, $\alpha + \beta < 1$ and for each $x, x_* \in X$, $x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq \alpha \frac{d(x_*, f(x_*))[1 + d(x, f(x))]}{1 + d(x, x_*)} + \beta d(x, x_*)$$

(2)' (Jaggi and Dass) – There exist numbers $\alpha, \beta \geq 0$, $\alpha + \beta < 1$ and for each $x, x_* \in X$, $x \neq x_*$, $x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq \alpha \frac{d(x, f(x)) d(x_*, f(x_*))}{d(x, f(x_*)) + d(x_*, f(x)) + d(x, x_*)} + \beta d(x, x_*)$$

(3)' (Gupta and Saxena)– There exist numbers a, b, $c \geq 0$, $a+b+c<1$ and for each $x, x_* \in X, x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq \frac{a[1 + d(x, f(x))] d(x_*, f(x_*))}{1 + d(x, x_*)} + \frac{b d(x, f(x)) d(x_*, f(x_*))}{d(x, x_*)} + c d(x, x_*)$$

(4)' (Jaggi) – There exist numbers $\alpha, \beta \geq 0$, $\alpha + \beta < 1$ and for each $x, x_* \in X$, $x \neq x_*$, $x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq \alpha \frac{d(x, f(x)) d(x_*, f(x_*))}{d(x, x_*)} + \beta d(x, x_*)$$

(5)' (Khan) – There exists a number k, $0 \leq k < 1$ and for each $x, x_* \in X$, $x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq k \frac{d(x, f(x)) d(x, f(x_*)) + d(x_*, f(x_*)).d(x_*, f(x))}{d(x, f(x_*)) + d(x_*, f(x))}$$

(6)' (Jain and Dixit) – There exist α_i, $\beta_i \geq 0$, $\alpha_1 + 2\alpha_3 + 2\alpha_4 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1$, $\alpha_2 + \beta_1 + \beta_4 + \beta_5 < 1$ and for each $x, x_* \in X$, $x \neq x_*$, $x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq \alpha_1 \frac{d(x, f(x)).d(x_*, f(x_*))}{d(x, x_*)} + \alpha_2 \frac{d(x, f(x_*)).d(x_*, f(x))}{d(x, x_*)} + \alpha_3 \frac{d(x_*, f(x)).d(x_*, f(x_*))}{d(x, x_*)}$$
$$+ \alpha_4 \frac{d(x, f(x)).d(x_*, f(x_*))}{d(x, x_*)} + \beta_1 d(x, x_*) + \beta_2 d(x, f(x)) + \beta_3 d(x_*, f(x_*)) + \beta_4 d(x, f(x_*))$$
$$+ \beta_5 d(x_*, f(x))$$

(7)' (Sharma and Bajaj) – There exist a number β, $0 < \beta < ½$ and for each $x, x_* \in X$, $x_* \in 0(x)$ such that

$$d(f(x), f(x_*)) \leq \beta \frac{d(x, f(x)).d(x, f(x_*))}{d(x, f(x)) + d(x, f(x_*))}$$

(8)' (Dass) – There exist numbers α_i , β_j >0 with $\alpha_1+\alpha_2+\alpha_3+\sum_{j=1}^{5}\beta_j<1$ for each positive integer m, and

for each x, $x_* \in$ X, x $\neq x_*$, $x_* \in 0$ (x) such that

$$d(f^m(x), f^m(x_*)) \le \alpha_1 \frac{d(x, f^m(x)).d(x_*, f^m(x_*))}{d(x,x_*)} + \alpha_2 \frac{d(x, f^m(x)).d(x_*, f^m(x))}{d(f^m(x), f^m(x_*))}$$
$$+\alpha_3 \frac{d(x, f^m(x_*)).d(x_*, f^m(x_*))}{d(f^m(x), f^m(x_*))} + \beta_1 d(x,x_*) + \beta_2 d(x, f^m(x))$$
$$+\beta_3 d(x_*, f^m(x_*)) + \beta_4 d(x, f^m(x_*)) + \beta_5 d(x_*, f^m(x))$$

(9)' (Pachpatte Thm.1) – There exists a number $q_1 \in$ (0,1), and for each x, $x_* \in$X, x $\neq x_*$, $x_* \in 0$ (x) such that

$$d(f(x), f(x_*)) \le q_1 \max\left\{d(x,x_*), \frac{d(x,f(x)).d(x_*, f(x_*))}{d(x,x_*)}, \frac{d(x,f(x_*)).d(x_*, f(x))}{d(x,x_*)}, \frac{d(x,f(x)).d(x,f(x_*))}{2d(x,x_*)}\right\}$$

(10)' (Pachpatte Thm.2) – There exists a number $q_2 \in$ (0,1), and for each x, $x_* \in$X, x $\neq x_*$,$x_* \in 0$(x) such that

$$\min\left\{d(f(x), f(x_*)), d(x, f(x)), d(x_*, f(x_*)), \frac{d(x,f(x)).d(x_*.f(x_*))}{d(x,x_*)}\right\} -$$
$$\min\left\{\frac{d(x,f(x_*)).d(x_*, f(x))}{d(x,x_*)}, \frac{d(x,f(x)).d(x,f(x_*))}{d(x,x_*)}\right\} \le q_2 d(x,x_*)$$

Now we prove the following results:

2.1 Theorem

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate which has the f. p. p. and let f:XxZ→XxZ be a mapping.

(a) If f is uniformly continuous such that for each z∈Z, $f_z \in (3)'$, then f has a fixed point.

(b) If X is locally compact, f is continuous such that for each z∈Z, $f_z \in (3)'$, then f has a fixed point.

Proof: We prove (a) and (b) simultaneously:

Step I: We define a sequence $\{t_n\}$ in X as follows:

For a fixed x_0 in X,

$$f_z^0(x_0) = t_0 = x_0, t_n(z) = f_z^n(x_0) = \pi_1 f(f_z^{n-1}(x_0), z); n \ge 1$$

For $f_z \in (3)'$ we have for x, $x_* \in$X, $x_* \in 0$(x) there exist a, b, c≥0 with a+ b+ c<1 such that

$$d(f_z(x), f_z(x_*)) \le \frac{a.[1+d(x, f_z(x))].d(x_*, f_z(x_*))}{1+d(x,x_*)} + \frac{b.d(x, f_z(x)).d(x_*, f_z(x_*))}{d(x,x_*)} + c.d(x,x_*)$$

Set $x_* = f_z(x)$ in the above inequality to obtain

$$d(f_z(x), f_z^2(x)) \le (a+b)d(f_z(x), f_z^2(x)) + c.d(x, f_z(x))$$

which implies that

$$d(f_z(x), f_z^2(x)) \le \left(\frac{c}{1-a-b}\right).d(x, f_z(x))$$

Now, set x=x*, then we have

$$d(f_z(x_*), f_z^2(x_*)) \leq \left(\frac{c}{1-a-b}\right).d(x_*, f_z(x_*))$$

Repeating above substitute we obtain

$$d(f_z^2(x), f_z^3(x)) \leq \left(\frac{c}{1-a-b}\right)^2.d(x, f_z(x))$$

Using induction, we get

$$d(f_z^n(x), f_z^{n+1}(x)) \leq \left(\frac{c}{1-a-b}\right)^n.d(x, f_z(x))$$

Finally set x=x_0, we get

$$d(t_n, t_{n+1}) \leq h^n.d(t_0, t_1), \quad where\ h = \left(\frac{c}{1-a-b}\right) < 1$$

Using triangle inequality, we find, for m > n

$$d(t_n, t_m) \leq d(t_n, t_{n+1}) + d(t_{n+1}, t_{n+2}) + ... + d(t_{m-1}, t_m)$$

$$\leq (h^n + h^{n+1} + ... + h^{m-1}).d(t_0, t_1)$$

$$= \frac{h^n(1-h^{m-n}).d(t_0, t_1)}{1-h} < \frac{h^n.d(t_0, t_1)}{1-h}$$

Since $h^n \to 0$ as $n \to \infty$, this inequality shows that $\{t_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists a point p_1 in X such that $t_n \to p_1$

Step II: we show that p_1 is a unique fixed point of f_z.

Since $f_z \in (3)'$. We have (taking x=t_n, x*=p_1)

$$d(f_z(t_n), f_z(p_1)) \leq \frac{a.[1 + d(t_n, f_z(t_n))]}{1 + d(t_n, p_1)} d(p_1, f_z(p_1)) + \frac{b.d(t_n, f_z(t_n)), d(p_1, f_z(p_1))}{d(t_n, p_1)} + c.d(t_n, p_1)$$

Since f_z is continuous, taking $n \to \infty$, we get

$d(p_1, f_z(p_1)) \leq a.\ d(p_1, f_z(p_1))$

which is possible only when $d(p_1, f_z(p_1))=0$ or $p_1=f_z(p_1)$, i.e., p_1 is a fixed point of f_z.

Suppose p_2 is another fixed point of f_z such that $p_1 \neq p_2$ then,

$$d(p_1, p_2) = d(f_z(p_1), f_z(p_2)) \leq \frac{a.[1 + d(p_1, f_z(p_1))]}{1 + d(p_1, p_2)} d(p_2, f_z(p_2)) + \frac{b.d(p_1, f_z(p_1)).d(p_2, f_z(p_2))}{d(p_1, p_2)}$$

$$+ c.d(p_1, p_2)$$

$$\Rightarrow d(p_1, p_2) \leq c.d(p_1, p_2)$$

which is a contradiction. Hence $p_1=p_2$ or p_1 is a unique fixed point of f_z.

Step III: Let a mapping F:Z→X be such that F(z)=p_1 is the unique fixed point of f_z. Now, let $z_0 \in Z$ and let $\{z_i\}_{i=1}^{\infty}$ be a sequence in Z which converges to z_0. Then by the hypothesis in (a), the sequence $\{f_{z_i}\}_{i=1}^{\infty}$ converges uniformly to f_{z0} and hence by the Theorem 1.1, the sequence $\{F(z_i)\}_{i=1}^{\infty}$ converges to F(z_0), under the assumptions of (b), we may apply Theorem 1.2, to conclude that the sequence $\{F(z_i)\}_{i=1}^{\infty}$ converges to F(z_0). Hence in either case this proves that F is continuous on Z. Also π_1f(F(z), z)=f_z(F(z))=F(z), because F(z) is a fixed point of f_z. Next, let G:Z→Z be defined by setting G(z)= π_2f(F(z),z). Then G is a continuous map of Z to itself. Since Z has the f.p.p., there exists a point $p \in Z$ such that G(p)=p, then the point (F(p), p) is such that π_1f(F(p), p) =F(p) and π_2f(F(p),p)= G(p) = p. Therefore (F(p), p) is a fixed point of f in XxZ.

2.2 Theorem

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate which has the f. p. p. and let f:XxZ→XxZ be a mapping.

(a) If f is uniformly continuous such that for each $z \in Z$, f_z satisfies any one of the conditions (2)', (5)', (6)', (7)', (8)', (9)' and (10)', then f has a fixed point.

(b) If (X, d) is locally compact, f is continuous such that for each $z \in Z$, f_z satisfies any one of the conditions (2)', (5)', (6)', (7)', (8)', (9)' and (10)', then f has a fixed point.

Proof: We prove (a) and (b) simultaneously:

We define a sequence $t_n(z)= t_n$ in X as follows:

For a fixed x_0 in X and any $z \in Z$,

$$f_z^0(x_0) = t_0, t_n = f_z^n(x_0) = \pi_1 f(f_z^{n-1}(x_0), z); n \geq 1$$

If f is such that $f_z \in (2)'$ and apply $x_* = f_z(x)$ then we have

$$d(f_z(x), f_z^2(x)) \leq \alpha . \frac{d(x, f_z(x)).d(f_z(x), f_z^2(x))}{d(x, f_z^2(x)) + d(f_z(x), f_z^2(x)) + d(x, f_z(x))} + \beta . d(x, f_z(x))$$
$$\leq \alpha . d(f_z(x), f_z^2(x)) + \beta . d(x, f_z(x))$$

or $d(f_z(x), f_z^2(x)) \leq \left(\frac{\beta}{1-\alpha}\right) d(x, f_z(x))$

Let $x=x_*$ in above inequality we have

$$d(f_z(x_*), f_z^2(x_*)) \leq \left(\frac{\beta}{1-\alpha}\right) d(x_*, f_z(x_*))$$

Again set $x_* = f_z(x)$, then we can obtain

$$d(f_z^2(x), f_z^3(x)) \leq \left(\frac{\beta}{1-\alpha}\right)^2 d(x, f_z(x))$$

By the induction we can write above relation as

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$$d(f_z^n(x), f_z^{n+1}(x)) \le \left(\frac{\beta}{1-\alpha}\right)^n d(x, f_z(x))$$

Finally set x=x_0, then we obtain

$$d(t_n, t_{n+1}) \le \left(\frac{\beta}{1-\alpha}\right)^n d(t_0, t_1)$$

…….(1)

Here we note that if the function f: XxZ→XxZ is such that $f_z \in (5)'$ then by using similar arguments, we can show that

$$d(t_n, t_{n+1}) \le k^n d(t_0, t_1)$$

…….(2)

Similarly, if f is such that $f_z \in (6)'$, then we can obtain, the condition

$$d(t_n, t_{n+1}) \le \left(\frac{\beta_1 + \beta_2 + \beta_4}{1-\alpha_1 - \beta_3 - \beta_4}\right)^n d(t_0, t_1)$$

…….(3)

Likewise, if f is such that $f_z \in (7)'$, then we obtain

$$d(t_n, t_{n+1}) \le \beta^n d(t_0, t_1)$$

…….(4)

Now, if f is such that $f_z \in (9)'$, then we can obtain

$$d(t_n, t_{n+1}) \le q_1^n d(t_0, t_1)$$

…….(5)

If f is such that $f_z \in (10)'$, then we can obtain

$$d(t_n, t_{n+1}) \le q_2^n d(t_0, t_1)$$

…….(6)

Finally, if the function f is such that $f_z \in (8)'$, then to obtain a condition of the type above, we proceed as follows:

Define $g_1 = f_z^m$, then we have

$$d(g_1(x), g_1(x_*)) \le \alpha_1 \frac{d(x, g_1(x)).d(x_*, g_1(x_*))}{d(x, x_*)} + \alpha_2 \frac{d(x, g_1(x)).d(x_*, g_1(x))}{d(g_1(x), g_1(x_*))}$$
$$+ \alpha_3 \frac{d(x, g_1(x_*)).d(x_*, g_1(x_*))}{d(g_1(x), g_1(x_*))} + \beta_1 d(x, x_*) + \beta_2 d(x, g_1(x))$$
$$+ \beta_3 d(x_*, g_1(x_*)) + \beta_4 d(x, g_1(x_*)) + \beta_5 d(x_*, g_1(x))$$

…….(7)

Using symmetry in (7), we have

$$d(g_1(x_*),g_1(x)) \le \alpha_1 \frac{d(x_*,g_1(x_*)).d(x,g_1(x))}{d(x_*,x)} + \alpha_2 \frac{d(x_*,g_1(x_*)).d(x,g_1(x_*))}{d(g_1(x_*),g_1(x))}$$
$$+\alpha_3 \frac{d(x_*,g_1(x)).d(x,g_1(x))}{d(g_1(x_*),g_1(x))} + \beta_1 d(x_*,x) + \beta_2 d(x_*,g_1(x_*))$$
$$+\beta_3 d(x,g_1(x)) + \beta_4 d(x_*,g_1(x)) + \beta_5 d(x,g_1(x_*))$$
…….(8)

Adding (7) and (8) above, we get

$$d(g_1(x),g_1(x_*)) \le \gamma_1 \frac{d(x,g_1(x)).d(x_*,g_1(x_*))}{d(x_*,x)}$$
$$+\gamma_2 \frac{[d(x,g_1(x)).d(x_*,g_1(x)) + d(x,g_1(x_*))d(x_*,g_1(x_*))]}{d(g_1(x),g_1(x_*))} + \gamma_3 d(x,x_*)$$
$$+\gamma_4[d(x,g_1(x)) + d(x_*,g_1(x_*))] + \gamma_5[d(x,g_1(x_*)) + d(x_*,g_1(x))]$$
….(9)

where, $\gamma_1 = \alpha_1, \quad \gamma_2 = \frac{\alpha_2+\alpha_3}{2}, \quad \gamma_3 = \beta_1, \quad \gamma_4 = \frac{\beta_2+\beta_3}{2}, \quad and \quad \gamma_5 = \frac{\beta_4+\beta_5}{2}$

with $\gamma_1 + 2\gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 = \alpha_1 + \alpha_2 + \alpha_3 + \sum_{i=1}^{5} \beta_i < 1$

In the equation (9), we apply similar procedure described above for the equation (1) and if mapping g_1 referred as f_z then we can obtain

$$d(t_n,t_{n+1}) \le \left(\frac{\gamma_2+\gamma_3+\gamma_4+\gamma_5}{1-\gamma_1-\gamma_2-\gamma_4-\gamma_5} \right)^n d(t_0,t_1)$$
…….(10)

Clearly according to conditions (2)', (5)', (6)', (7)', (9)', (10)' and (8)' we obtain equations (1), (2), (3), (4), (5), (6) and (10) respectively. However, in each of these cases we see that $\{t_n\}$ is a Cauchy sequence in X. However, by the completeness of X, there is a point p_1 in X such that t_n converges to p_1. We can easily see that p_1 is a unique fixed point of f_z. By the help of step-III of the above theorem 2.1, we can conclude the theorem 2.2.

III. CONCLUSION

We observe that condition (1)', (4)' are stronger than (3)', therefore the above Theorem 2.1 has two corollaries corresponding to each of these two conditions. We also observe that condition (4)' is stronger than conditions (6)' and (8)' therefore the Theorem 2.2 has one corollary corresponding to (4)'. This paper is extension of Nadler's result according to contractive conditions of Rhoades [7].

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