International Journal of Advanced Technology in Engineering and Science
Vol. No.5, Issue No. 01, January 2017
www.ijates.com
ijates
ISSN 2348 - 7550

GENERATING OR CONTROLLING CHAOS IN NONLINEAR COURNOT DUOPOLY

Bharti Kapoor[1], Dr.Amanpreet Singh[2]

[1]*Research Scholar Punjab Technical University, Jalandhar, Punjab(India)*

[2]*Post Graduate Department of Mathematics*

SGTB Khalsa College Anandpur Sahib, Ropar Punjab140118 (India)

ABSTRACT

This study with the help of non-linear Cournot Duopoly model shows that duopoly market can be chaotic because of non-linearity. It establishes conditions for the stability of chaotic market. The study considers the conditions for generating chaos and controlling chaos from the perspectives of both the firms. It explains how adaptive expectations can be used due to inconsistency of naïve expectations with the help of autocorrelation coefficient and then explains the method of controlling chaos.

Keywords: Autocorrelation coefficient, Period doubling, Adaptive control method

I. INTRODUCTION

Oligopoly is a market system which is controlled by few firms i.e. only few firms dominate the market. Cournot was the first who introduced the concept of Oligopoly. Duopoly is the sub-case of oligopoly. In typical Cournot duopoly, there were only two firms in the market. Each firm wants to maximize profit and there was not collusive situation and chaos in the market. But actually one may observe chaos in oligopoly and Duopoly market under certain conditions. To know whether there is chaos in the system or not, we need Cournot model. Cournot duopoly model may be linear as well as non-linear. Nonlinearities may be due to several reasons like demand function is non-linear, due to non-linearties of cost structure, Heterogeneity in strategies, effect of each firm's own and each other's R& D, differentiated goods etc. When non-linearites are very strong then it generates complex dynamics in which we observe chaos. In this paper, we will construct Non-linear Cournot Duopoly model. Moreover, we will study factors which are responsible for generating chaos and controlling chaos. It is yet to be determined that generating chaos is more profitable or controlling chaos. Study of controlling chaos has been given by *Kople, Kass, Beta, Mendas and Mendas.* In recent studies it has been revealed that if goods are complementary then generating chaos is more profitable from long-run prospective in two dimensional output adjustment process. We will see that chaos created by cournot competition is in double bind from long run perspective, Firms with higher marginal cost prefer chaos in market and firms with lower marginal cost prefer stable market. So, if we decide to control chaos or generate chaos, and see from long run point of view then only one of the players out of two will be beneficial at a time.

According to Puu's model price p is reciprocal to total demand D,

$$p = \frac{1}{D}$$

Two firms *firm 1* and *firm 2*, produce amount of goods *a* and *b* respectively with constant marginal costs C_1

and C_2 respectively. $D = x + y$.

Profit is given by

$$\pi_1 = \frac{a}{a + b^e} - c_1 a \qquad (1)$$

And $$\pi_2 = \frac{b}{a^e + b} - c_2 b \qquad (2)$$

Where *"e"* denotes the expected value. For Firm *1*, *a* is under control and it takes expected value of *b* , and same for firm *2*.

In order to find reaction functions, we take partial derivatives of equation (1) and (2) and substitute equal to zero.

Taking partial derivatives w.r.t. *a* and *b* respectively, we get

$$\frac{\partial \pi_1}{\partial a} = \frac{b}{(a+b)^2} - c_1$$

And $$\frac{\partial \pi_2}{\partial b} = \frac{a}{(a+b)^2} - c_2$$

Taking these partial derivatives above derived equal to zero, we get:

$$b = c_1 (a+b)^2$$

$$\Rightarrow \quad a_{t+1} = \sqrt{\frac{b_t}{c_1}} - b_t \quad = f(b_t) \qquad (3)$$

Similarly, taking eq. (4) equal to zero, we get:

$$b_{t+1} = \sqrt{\frac{a_t}{c_2}} - a_t, \quad = g(a_t) \qquad (4)$$

which shows that expected value of output at time '*t+1*' of *firm1* is a function output of *firm* 2 at time *'t'*. So, equations (3) and (4) are reaction functions.

For these values of *a* and *b*, we will get maximum profit as second order derivatives will be negative.

Now, we calculate zeroes of reaction function. For that put equation (3) to zero, we get

$$\sqrt{\frac{b}{c_1}} - b \ = 0$$

$$\Rightarrow b^2 = \frac{b}{c_1}$$

$$\Rightarrow b\left(b \ - \frac{1}{c_1}\right) = 0$$

$$\Rightarrow b \ = \frac{1}{c_1}$$

So, $b = \frac{1}{c_1}$, Similarly, $a = \frac{1}{c_2}$

In order to find maximum value of output , differentiate (3) w.r.t. $'b'$, so that

$$\frac{df}{db} = \frac{1}{2\sqrt{c_1}\sqrt{b}} - 1$$

$$Now, \frac{df}{db} = 0$$

$$\Rightarrow b = \frac{1}{4c_1} \qquad (5)$$

Similarly, $a = \frac{1}{4c_2}$ (6)

At these values of a and b , $\frac{d^2 f}{db^2} < 0$ and $\frac{d^2 f}{da^2} < 0$ respectively. So, these values are the maximum values of *a* and *b* respectively and Domain should be restricted to the interval $\left[0, \frac{1}{c_1}\right]$.

Also Cournot equilibrium is attained when (3) and (4) hold simultaneously.

Again from (3) and (4), we get

$$a + b = \sqrt{\frac{b}{c_1}} \quad \text{and} \quad a + b = \sqrt{\frac{a}{c_2}}$$

$$\Rightarrow \sqrt{\frac{b}{c_1}} = \sqrt{\frac{a}{c_2}}$$

$$\Rightarrow b = \frac{c_1 a}{c_2}$$

Substitute this value in (3), we get

$$a = \sqrt{\frac{a}{c_2}} - \frac{c_1 a}{c_2}$$

$$\Rightarrow a\left(\frac{c_1 + c_2}{c_2}\right) = \sqrt{\frac{a}{c_2}}$$

$$\Rightarrow a = \frac{c_2}{(c_1 + c_2)^2}$$

And $b = \dfrac{c_1}{(c_1 + c_2)^2}$ (7)

When we substitute these values of a and b in

$$f^{'}(a) = \frac{1}{2}\sqrt{\frac{1}{c_1 b}} - 1$$

and

$$g^{'}(b) = \frac{1}{2}\sqrt{\frac{1}{c_2 a}} - 1$$

We get derivatives in Cournot equilibrium point,

$\dfrac{1}{2}\dfrac{c_2 - c_1}{c_1}$ and $\dfrac{1}{2}\dfrac{c_1 - c_2}{c_2}$ (8)

The loss of stability for the fixed point occurs when Product of Cournot equilibrium point so obtained in equation (6) equal to -1

$$i.e.\ \frac{1}{2}\frac{c_2 - c_1}{c_1} \cdot \frac{1}{2}\frac{c_1 - c_2}{c_2} = -1$$

Solving this for $\dfrac{c_1}{c_2}$ or $\dfrac{c_2}{c_1}$, we get $3 \pm 2\sqrt{2}$

Then stability of fixed point occurs , whenever this ratio of the unit costs of two firms is within the interval $\left(3 - 2\sqrt{2}, 3 + 2\sqrt{2}\right)$ (9)

But if this ratio falls outside this interval then we have period doubling. Again, if we substitute the maximum value of $a = \dfrac{1}{4c_2}$ from eq.(6) in (4), we get the reaction function of the other firm as below:

$b = \dfrac{1}{\sqrt{4c_1 c_2}} - \dfrac{1}{4c_1}$, but this value must be less than $\dfrac{1}{c_1}$ which is zero of equation (3),otherwise the model will explode.

$$\text{i.e. } \frac{1}{\sqrt{4c_1 c_2}} - \frac{1}{4c_1} < \frac{1}{c_1}$$

$$\Rightarrow \frac{1}{\sqrt{4c_1c_2}} < \frac{5}{4c_1}$$

$$\Rightarrow \frac{1}{4c_1c_2} < \frac{25}{16c_1^2}$$

$$\Rightarrow \frac{c_1}{c_2} < \frac{25}{4}$$

Similarly taking other reaction function we will have the condition

$$\frac{c_1}{c_2} > \frac{4}{25}$$

So, we get the upper and lower bound of marginal c_2 in term of marginal cost c_1 ,

$$\frac{4}{25} \leq \frac{c_2}{c_1} \leq \frac{25}{4} \qquad (10)$$

As we are focused in controlling chaos, we first obtain the condition in which Cournot point is locally unstable. Let us suppose that $c_1 \neq c_2$.Considering economically feasible production level in (10) and stable interval in (9),we find that unstable condition can be obtained by the ratio of marginal production costs and taking either $a > b$ or $a < b$.

Under the assumption that $b > a$ and $3+2\sqrt{2} < \frac{c_2}{c_1} \leq \frac{25}{4}$,the Cournot point is unstable so that trajectories starting from any point of a neighbourhood of Cournot point move away ,come back to neighbourhood soon or later but move away again. So, the Dynamic process does not converge to Cournot point but keep fluctuating within limited region.

II. FEATURES OF CHAOTIC DYNAMICS

Sensitivity to the initial conditions and irregularity of trajectory are two statistical properties of chaotic dynamics. First property means that a small change in initial condition may result in different behavior of chaotic trajectory. For this reason instead of checking the behavior of individual trajectory, we investigate the long run average behavior of trajectory. Second property shows that it is not easy to predict the values of variables along chaotic trajectory.

III. BEHAVIOR FOR GREATER PERIOD

As mentioned above, chaotic trajectories are sensitive to initial conditions. So, two trajectories starting from similar initial conditions will behave in different and complicated way. We focus to find long run average behavior of such chaotic dynamics. Under weak mathematical conditions chaotic trajectories may converge to a stable density function. If it is possible to find the explicit form of such density function, one can investigate long-run average behavior of chaotic trajectories. Because of difficulty in finding explicit form of density function, one can numerically calculate the long run average behavior of chaotic trajectories. On calculating numerically we observe when Cournot point is unstable, the long-run average profit of the efficient firm is less than the Cournot profit while the long-run average profit of inefficient firm is more than Cournot profit. Here efficient firm means the firm whose marginal cost of production is less.

IV. AUTO CORRELATION

We use naïve expectations in eq.(3) and (4) which means that firms expect rival firms to have same output as in previous period, but naïve expectations may lead to systematic error, which means firms are making wrong expectations along chaotic trajectories. Actually chaotic fluctuations may be due to the randomness of the stochastic process if autocorrelation of output is zero at all time lags. Also under naïve expectations prices are correlated with past prices if the supply function is monotonic and uncorrelated if the supply function is non-monotonic. We show that the autocorrelation coefficient may be non-zero even if the dynamic process is chaotic.

Expectation errors are given by

$$e_t = a_t - a_t^e$$

The autocorrelation coefficients μ_k of expectation errors are defined as

$$\mu_k = \frac{\alpha_k}{\alpha_0}, \quad -1 \le \mu_k \le 1$$

With $\bar{e} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} e_t$

$$\alpha_k = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\left(e_t - \bar{e}\right)\left(e_{t+k} - \bar{e}\right) \qquad k \ge 0$$

An expectation may be consistent, weakly consistent or inconsistent. An expectation is consistent if autocorrelation coefficient of expectation error is zero for all $k \geq 1$, weakly consistent if it is zero for all $k \geq K$, Where $K \geq 2$ and inconsistent if it is not weakly consistent. Now expectations are consistent means irregularity of chaotic trajectories are due to randomness. So there is no need to change expectation formations. i.e. firms can use naïve expectations. While inconsistent expectations means that chaotic fluctuations are different from random fluctuations and therefore firms can alter their expectation formation accordingly. After investigating with the help of numerical approximation with suitable initial conditions and parameter values,we find that naïve expectations are inconsistent as autocorrelation coefficient so obtained does not satisfy the condition of consistent or weakly consistent.

V. LIMITING LOSS AND PROFIT

As mentioned above with the help of expectation error that naïve expectations are inconsistent. Therefore, duopolists shift to adaptive expectations from naïve expectations. For this they use weighted average of current period expectation output of the competitor and actual output of the current period. Adaptive expectation can be considered generalization of Naïve expectations. So, Equations (3) and (4) changes to

$$\begin{aligned} a_{t+1} &= (1-\lambda_x)a_t + \lambda_x f(b_t) \\ b_{t+1} &= \lambda_y g(a_t) + (1-\lambda_y)b_t \end{aligned} \qquad (12)$$

Where λ_x and λ_y are adjustment parameters of *firm 1* and *firm 2* respectively. As for $\lambda_x = \lambda_y = 0$, there will be no dynamics and for $\lambda_x = \lambda_y = 1$ equation (12) becomes same as equation (3) and (4). So, we assume the values of these adjustment parameters between 0 and 1. It can be checked that fixed points of equation (3) ,(4) and (12) are same. Jacobi matrix of adaptive process (12) can be calculated at cournot point and is given by

$$J = \begin{bmatrix} 1-\lambda_x & \lambda_x\left(\dfrac{c_2-c_1}{2c_1}\right) \\ \lambda_y\left(\dfrac{c_1-c_2}{2c_2}\right) & 1-\lambda_y \end{bmatrix}$$

tr J=2-($\lambda_x + \lambda_y$) and det J= $(1-\lambda_x)(1-\lambda_y) + \lambda_x\lambda_y \dfrac{(c_1-c_2)^2}{4c_1c_2}$ (13)

We know that for the stability of equilibrium point, both eigen values must have modulus less than one. So, Polynomial $\lambda^2 - tr\ J\,\lambda + \det J$ has roots less than unity in absolute value if and only if

(a) $\det J < 1$

(b) $\det J > tr\ J - 1$

(c) $\det J > -tr\ J - 1$ (14)

which are conditions of stability of equilibria.

From first condition, using (13) we get

$$1-(\lambda_x+\lambda_y)+\lambda_x\lambda_y+\lambda_x\lambda_y\frac{(c_1-c_2)^2}{4c_1c_2}<1$$

$$\Rightarrow 1-(\lambda_x+\lambda_y)+\lambda_x\lambda_y+\lambda_x\lambda_y\frac{c_1^2+c_2^2-2c_1c_2}{4c_1c_2}<1$$

$$\Rightarrow 1-(\lambda_x+\lambda_y)+\lambda_x\lambda_y+\lambda_x\lambda_y c_1^2\frac{\left(1+\left(\dfrac{c_2}{c_1}\right)^2-2\dfrac{c_2}{c_1}\right)}{4c_1c_2}<1$$

$$\Rightarrow 1-(\lambda_x+\lambda_y)+\lambda_x\lambda_y+\lambda_x\lambda_y\frac{(1+t^2-2t)}{4t}<1 \quad \text{where } t=\frac{c_2}{c_1}$$

$$\Rightarrow 4t-4t(\lambda_x+\lambda_y)+4t\lambda_x\lambda_y+\lambda_x\lambda_y(1+t^2-2t)<4t$$

$$\Rightarrow -4t(\lambda_x+\lambda_y)+4t\lambda_x\lambda_y+\lambda_x\lambda_y(1+t^2-2t)<0$$

$$\Rightarrow [-4t+4t\lambda_x+\lambda_x(1+t^2-2t)]\lambda_y<4t\lambda_x$$

$$\Rightarrow [-4t+4t\lambda_x+\lambda_x(1+t^2-2t)]\lambda_y<4t\lambda_x$$

$$\Rightarrow \lambda_y<\frac{4t\lambda_x}{(1+t)^2\lambda_x-4t} \qquad (15)$$

Similarly, from second and third condition of (14), we get

$$\frac{(1+t)^2}{4t}\lambda_x\lambda_y>0 \qquad (16)$$

and

$$2(2-\lambda_x-\lambda_y)+\frac{(1+t)^2}{4t}\lambda_x\lambda_y>0 \text{ respectively.} \qquad (17)$$

Now (16) and (17) are always true, as the values of adjustment parameters λ_x and λ_y lie between 0 and 1.This mean that iterative dynamics will not diverge or create period-doubling bifurcation. Using adaptive control methods one can stabilize chaotic market using sufficiently small value of adjustment parameter.

VI. CONCLUSION

1) In duopoly market out of two firms, firm which is efficient rules the market at cournot point
2) For stability of cournot point the ratio of marginal costs of both the firms must lie within interval $(3-2\sqrt{2},3+2\sqrt{2})$
3) A situation of chaos occurs when the ratio of marginal costs lie between $\left(3+2\sqrt{2},\frac{25}{4}\right]$
4) From long run average perspective, inefficient firm is more profitable when there is chaos in market. So, an efficient firm prefers stable market and inefficient firm prefers chaotic market.
5) Chaos in market can be controlled with the help of certain methods. Adaptive control method is one such method.

References

1) Huang, W-H., "Theory of Adaptive Adjustment," Discrete Dynamics in Nature and Society,5,247-263,2001.
2) Romeiras, F. J., C. Grebogi, E. Ott, and W.P. Dayawana, "Controlling Chaotic Dynamical Systems,"Physica D,58, 165-192, 1992.

3) Sakai, H., and H. Tokumaru, "Autocorrelations of a Certain Chaos," IEEE Transactions on Acoustics, Speech and Signal Processing,ASSP-28, 588-590, 1980.

4) Ueta, T., and Kawakami, H., "Composite Dynamic System for Controlling Chaos, "IEICE Trans. Fundamentals, E78-A (6), 708-714, 1995.

5) T. Puu and I. Sushko, Eds.,Oligopoly Dynamics, Springer, Berlin, Germany,2002.

6) T. Puu, Attractors, Bifurcations, & Chaos. Nonlinear Phenomena in Economics, Springer, Berlin, Germany,2nd edition, 2003