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SOME RELATED FIXED POINT THEOREMS FOR FOUR MENGER SPACES

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ABSTRACT

In the present paper we extend some related fixed point theorems due to Gupta [4] for four Menger spaces. The results of Gupta [4] are the special cases of the results established in this paper.

Keywords: Probabilistic metric space, Menger space, common fixed point.

AMS Subject Classification: Primary 47H10, Secondary 54H25

I. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [7]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [10] studied this concept and gave some fundamental results on this space.

Fisher [2] initiated the study of existence of related fixed point for two mappings on two metric spaces. Later Fisher and Murthy [3] proved some related fixed point theorems for two pairs of mappings on two metric spaces. Also some related fixed point theorems on metric spaces have been established by Namdeo and Fisher [9], Kikina and Kikina [6], Jain, Sahu and Fisher [5] and Ansari, Sharma and Garg [1]. Recently Gupta [4] established some related fixed point theorems for four mappings on four metric spaces.

In this paper, we extend and generalize the result of Gupta [4] to four Menger spaces and established some related fixed point theorems.

II. PRELIMINARIES

Definition 2.1. [8] A mapping $\mathcal{F} : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with

$\inf \{ F(t) \mid t \in R \} = 0$ and $\sup \{ F(t) \mid t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by $H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$.

Definition 2.2. [8] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions

(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a, d \ge b$,

(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

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Definition 2.3. [8] A *probabilistic metric space (PM-space)* is an ordered pair (X, $\mathcal{F}$) consisting of a non-empty set X and a function $\mathcal{F}: X \times X \rightarrow L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at (u, v) $\in X \times X$ is represented by $F_{u, v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$, for all u,v,w $\in$ X and x, y > 0.

Definition 2.4. [8] A *Menger space* is a triplet (X, $\mathcal{F}$, t) where (X, $\mathcal{F}$) is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \geq t\{F_{u, v}(x), F_{v, w}(y)\}$, for all u, v, w $\in$ X, x, y $\geq$ 0.

Definition 2.5. [8] A sequence $\{x_n\}$ in a Menger space (X, $\mathcal{F}$, t) is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \quad \text{for all } n \geq M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x_m}(\varepsilon) > 1- \lambda \quad \text{for all } m, n \geq M(\varepsilon, \lambda).$$

A Menger PM-space (X, $\mathcal{F}$, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.
A complete metric space can be treated as a complete Menger space in the following way:

Proposition 2.1. [8] If (X, d) is a metric space then the metric d induces mappings $\mathcal{F}: X \times X \rightarrow L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, p, q $\in$ X, where

$$H(k) = 0, \quad \text{for } k \leq 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0,1] \times [0,1] \rightarrow [0,1]$ is defined by $t(a, b) = \min\{a, b\}$. Then (X, $\mathcal{F}$, t) is a Menger space. It is complete if (X, d) is complete.
The space (X, $\mathcal{F}$, t) so obtained is called the *induced Menger space.*

Lemma 2.1. [8] Let (X, $\mathcal{F}$, t) be a Menger space. If there exists a constant $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t)$$

for all t > 0 with fixed x, y $\in$ X then x = y.

III. MAIN RESULT

Theorem 3.1. Let (X, F, *), (Y, G, *) and (Z, H, *) be the complete Menger spaces, where * is a continuous t-norm (i.e. min t-norm). Let T be a continuous mapping of X into Y, S be a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities :

(3.1) $F_{RSTx,RSTx'}(kt) \geq \min\{F_{x,x'}(t), F_{x,RSTx}(t), F_{x',RSTx'}(t), G_{Tx,Tx'}(t), H_{STx, STx'}(t)\}$

(3.2) $G_{TRSy,TRSy'}(kt) \geq \min\{G_{y,y'}(t), G_{y,TRSy}(t), G_{y',TRSy'}(t), H_{Sy,Sy'}(t), F_{RSy, RSy'}(t)\}$

(3.3) $H_{STRz,STRz'}(kt) \geq \min\{H_{z,z'}(t), H_{z,STRz}(t), H_{z',SRTz'}(t), F_{Rz,Rz'}(t), G_{TRz, TRz'}(t)\}$

for all x, x' in X, y, y' in Y and z, z' in Z where $0 \leq k \leq 1$.

Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further Tu = v, Sv = w and Rw = u.

Proof. Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively by

$$x_n = (RST)^n x_0,\ y_n = Tx_{n-1},\ z_n = Sy_n \text{ for } n = 0, 1, 2, \ldots .$$

Applying inequality (3.2), we have

$$G_{y_n,y_{n+1}}(kt) = G_{TRSy_{n-1}, TRSy_n}(t) \quad (3.4)$$

$$\geq \min\{G_{y_{n-1},y_n}(t), G_{y_n, y_{n+1}}(t), H_{z_{n-1}, z_n}(t), F_{x_{n-1}, x_n}(t)\}$$

$$= \min\{F_{x_{n-1},x_n}(t), G_{y_{n-1}, y_n}(t), H_{z_{n-1}, z_n}(t)\}$$

Using inequality (3.3), we have

$$H_{z_n,z_{n+1}}(kt) = H_{STRz_{n-1}, STRz_n}(t) \quad (3.5)$$

$$\geq \min\{H_{z_{n-1},z_n}(t), H_{z_n,z_{n+1}}(t), F_{x_{n-1},x_n}(t), G_{y_n, y_{n+1}}(t)\}$$

$$= \min\{F_{x_{n-1},x_n}(t), G_{y_{n-1}, y_n}(t), H_{z_{n-1}, z_n}(t)\}$$

on using inequality (3.4).

Using inequality (3.1), we have

$$F_{x_n,x_{n+1}}(kt) = F_{RSTx_{n-1}, RSTx_n}(t) \quad (3.6)$$

$$\geq \min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t), G_{y_n,y_{n+1}}(t), H_{z_n, z_{n+1}}(t)\}$$

$$= \min\{F_{x_{n-1},x_n}(t), G_{y_{n-1}, y_n}(t), H_{z_{n-1}, z_n}(t)\}.$$

on using inequalities (3.4) and (3.5).

It now follows easily by induction on using inequalities (3.4), (3.5) and (3.6) that

$$F_{x_n,x_{n+1}}(k^{n-1}t) \geq \min\{F_{x_1,x_2}(t), G_{y_1, y_2}(t), H_{z_1, z_2}(t)\}$$

$$G_{y_n,y_{n+1}}(k^{n-1}t) \geq \min\{F_{x_1,x_2}(t), G_{y_1, y_2}(t), H_{z_1, z_2}(t)\}$$

$$H_{z_n,z_{n+1}}(k^{n-1}t) \geq \min\{F_{x_1,x_2}(t), G_{y_1, y_2}(t), H_{z_1, z_2}(t)\}.$$

Since k < 1, it follows that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are Cauchy sequences with limits u, v and w in X, Y and Z respectively.

Since T and S are continuous, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tu = v,$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} Sy_n = Sv = w.$$

Using inequality (3.1) again, we have

$$F_{RSTu,x_n}(kt) = F_{RSTu,x_{n-1}}(t)$$

$$\geq \min\{F_{u,x_{n-1}}(t), F_{u,RSTu}(t), F_{x_{n-1},x_n}(t), G_{Tu,Tx_{n-1}}(t), H_{STu,\,STx_{n-1}}(t)\}.$$

Since T and S are continuous, it follows on letting $n \to \infty$ that

$$F_{RSTu,\,u}(kt) \geq F_{u,RSTu}(t).$$

Thus, RSTu = u, since k < 1 and so u is fixed point of RST.

We now have

$$TRSv = TRSTu = Tu = v \text{ and so}$$

$$SRTw = STRv = Sv = w.$$

Hence, v and w are fixed points of TRS and STR respectively.

Now, we prove the uniqueness of the fixed point u.

Suppose that RST has a second fixed point u'.

Then using inequality (3.1), we have

$$F_{u,u'}(kt) = F_{RSTu,\,RSTu'}(kt)$$

$$\geq \min\{F_{u,u'}(t), F_{u,RSTu}(t), F_{u',RSTu'}(t), G_{Tu,Tu'}(t), H_{STu,\,STu'}(t)\}$$

$$= \min\{G_{Tu,Tu'}(t), H_{STu,STu'}(t)\}.$$

Further using inequality (3.2), we have

$$G_{Tu,Tu'}(kt) = G_{TRSTu,\,TRSTu'}(kt)$$

$$\geq \min\{G_{Tu,Tu'}(t), G_{Tu,TRSTu}(t), G_{Tu',TRSTu'}(t), H_{STu,\,STu'}(t), F_{RSTu,RSTu'}(t)\}$$

$$= \min\{F_{u,u'}(t), H_{STu,STu'}(t)\}.$$

Hence, we have

$$F_{u,u'}(kt) \geq H_{STu,STu'}(t).$$

And finally on using inequality (3.3), we now have

$$F_{u,u'}(kt) \geq H_{STu,\,STu'}(t) = H_{STRSTu,\,STRSTu'}(t)$$

$$\geq k^2\max\{H_{STu,STu'}(t), H_{STu,STRSTu}(t), H_{STu',STRSTu'}(t), F_{RSTu,\,RSTu'}(t), G_{TRSTu,TRSTu'}(t)\}$$

$$= k^2\max\{H_{STu,STu'}(t), F_{u,u'}(t), G_{Tu,Tu'}(t)\}.$$

Since k < 1, it follows that u = u' and the uniqueness of u follows.

Similarly, it can be proved that v is the unique fixed point of the TRS and w is the unique fixed point of STR.

We finally prove that we also have

$$4Rw = R(STRw) = RST(Rw)$$

and so Rw is the fixed point of RST. Since u is the unique fixed point of RST, it follows that Rw = u.

This completes the proof of the theorem.

Theorem 3.2. Let $(X_1, F_1, *)$, $(X_2, F_2, *)$, $(X_3, F_3, *)$ and $(X_4, F_4, *)$ be the complete Menger spaces, where * is a continuous t-norm (i.e. min t-norm). If T_1 is a continuous continuous mapping of X_1 into X_2, T_2 be a continuous mapping of X_2 into X_3, T_3 be a continuous mapping of X_3 into X_4 and T_4 be a mapping of X_4 into X_1 satisfying the inequalities :

(3.7) $$F_{1\,T_4T_3T_2T_1x_1,\,T_4T_3T_2T_1x_1'}(kt) \ge \min\{F_{1\,x_1,x_1'}(t), F_{1\,x_1,T_4T_3T_2T_1x_1}(t), F_{1\,x_1',T_4T_3T_2T_1x_1'}(t),$$
$$F_{2\,T_1x_1,T_1x_1'}(t), F_{3\,T_2T_1x_1,T_2T_1x_1'}(t), F_{4\,T_3T_2T_1x_1,T_3T_2T_1x_1'}(t)\}$$

(3.8) $$F_{2\,T_4T_3T_2T_1x_2,\,T_4T_3T_2T_1x_2'}(kt) \ge \min\{F_{2\,x_2,x_2'}(t), F_{2\,x_2,T_1T_4T_3T_2x_2}(t), F_{2\,x_2',T_1T_4T_3T_2x_2'}(t),$$
$$F_{3\,T_2x_2,T_2x_2'}(t), F_{4\,T_3T_2x_2,T_3T_2x_2'}(t), F_{1\,T_4T_3T_2x_2,T_4T_3T_2x_2'}(t)\}$$

(3.9) $$F_{3\,T_2T_1T_4T_3x_3,\,T_2T_1T_4T_3x_3'}(kt) \ge \min\{F_{3\,x_3,x_3'}(t), F_{3\,x_3,T_2T_1T_4T_3x_3}(t), F_{3\,x_3',T_2T_1T_4T_3x_3'}(t),$$
$$F_{4\,T_3x_3,T_3x_3'}(t), F_{1\,T_4T_3x_3,T_4T_3x_3'}(t), F_{2\,T_1T_4T_3x_3,T_1T_4T_3x_3'}(t)\}$$

(3.10) $$F_{4\,T_3T_2T_1T_4x_4,\,T_3T_2T_1T_4x_4'}(kt) \ge \min\{F_{4\,x_4,x_4'}(t), F_{4\,x_4,T_3T_2T_1T_4x_4}(t), F_{4\,x_4',T_3T_2T_1T_4x_4'}(t),$$
$$F_{1\,T_4x_4,T_4x_4'}(t), F_{2\,T_1T_4x_4,T_1T_4x_4'}(t), F_{3\,T_2T_1T_4x_4,T_2T_1T_4x_4'}(t)\}$$

for all $x_1, x_1' \in X_1$, $x_2, x_2' \in X_2$, $x_3, x_3' \in X_3$, $x_4, x_4' \in X_4$, where $0 \le k \le 1$.

Then $T_4T_3T_2T_1$ has a unique fixed point α_1 in X_1, $T_1T_4T_3T_2$ has a unique fixed point α_2 in X_2, $T_2T_1T_4T_3$ has a unique fixed point α_3 in X_3 and $T_3T_2T_1T_4$ has a unique fixed point α_4 in X_4.

Proof. Let x_1^0 be an arbitrary point in X_1. Define the sequences $\{x_n^1\}$, $\{x_n^2\}$, $\{x_n^3\}$ and $\{x_n^4\}$ in X_1, X_2, X_3 and X_4 respectively by

$$(T_4T_3T_2T_1)^n x_1^0 = x_n^1$$

$$T_1x_{n-1}^1 = x_n^2$$

$$T_2x_n^2 = x_n^3$$

$$T_3x_n^3 = x_n^4$$

$$T_4x_n^4 = x_n^1 \quad \text{for } n = 1, 2, 3, \dots .$$

Now applying inequality (3.8), we get

$$F_{2\,x_n^2,x_{n+1}^2}(kt) = F_{2\,T_1T_4T_3T_2x_{n-1}^2,\,T_1T_4T_3T_2x_n^2}(t)$$

$$\geq \min\{F_{2\,x_{n-1}^2,x_n^2}(t), F_{2\,x_{n-1}^2,T_1T_4T_3T_2x_{n-1}^2}(t), F_{2\,x_n^2,T_1T_4T_3T_2x_n^2}(t),$$

$$F_{3\,T_2x_{n-1}^2,T_2x_n^2}(t), F_{4\,T_3T_2x_{n-1}^2,T_3T_2x_n^2}(t), F_{1\,T_4T_3T_2x_{n-1}^2,T_4T_3T_2x_n^2}(t)\}$$

$$F_{2x_n^2,x_{n+1}^2}(kt) \geq \min\{F_{2\,x_{n-1}^2,x_n^2}(t), F_{2x_{n-1}^2,x_n^2}(t), F_{2\,x_n^2,x_{n+1}^2}(t),\ F_{3\,x_{n-1}^3,x_n^3}(t), F_{4\,x_{n-1}^4,x_n^4}(t), F_{1x_{n-1}^1,x_n^1}(t)\}$$

$$F_{2x_n^2,x_{n+1}^2}(kt) \geq \min\{F_{1\,x_{n-1}^1,x_n^2}(t), F_{2x_{n-1}^2,x_n^2}(t), F_{3\,x_{n-1}^3,x_n^3}(t),\ .\ F_{3\,x_{n-1}^3,x_n^3}(t), F_{4\,x_{n-1}^4,x_n^4}(t)\}\ . \tag{3.11}$$

Using inequality (3.9), we get

$$F_{3x_n^3,x_{n+1}^3}(kt) = F_{3\,T_2T_1T_4T_3x_{n-1}^3,\,T_2T_1T_4T_3x_n^3}(t)$$

$$\geq \min\{F_{3\,x_{n-1}^3,x_n^3}(t), F_{3x_{n-1}^3,T_2T_1T_4T_3x_{n-1}^3}(t), F_{3\,x_n^3,T_2T_1T_4T_3x_n^3}(t),$$

$$F_{4\,T_3x_{n-1}^3,T_3x_n^3}(t), F_{1\,T_4T_3x_{n-1}^3,T_4T_3x_n^3}(t), F_{2\,T_1T_4T_3x_{n-1}^3,T_1T_4T_3x_n^3}(t)\} \tag{3.12}$$

$$F_{3x_n^3,x_{n+1}^3}(kt) \geq \min\{F_{3\,x_{n-1}^3,x_n^3}(t), F_{3x_{n-1}^3,x_n^3}(t), F_{3\,x_n^3,x_{n+1}^3}(t),\ F_{4\,x_{n-1}^4,x_n^4}(t), F_{1\,x_{n-1}^1,x_n^1}(t), F_{2x_{n-1}^2,x_n^2}(t)\}$$

$$F_{3x_n^3,x_{n+1}^3}(kt) \geq \min\{F_{1\,x_{n-1}^1,x_n^1}(t), F_{2x_{n-1}^2,x_n^2}(t), F_{3\,x_{n-1}^3,x_n^3}(t),\ F_{4\,x_{n-1}^4,x_n^4}(t)\}\ .$$

Using inequality (3.10), we get

$$F_{4x_n^4,x_{n+1}^4}(kt) = F_{4\,T_3T_2T_1T_4x_{n-1}^4,\,T_3T_2T_1T_4x_n^4}(t)$$

$$\geq \min\{F_{4\,x_{n-1}^4,x_n^4}(t), F_{4x_{n-1}^4,T_3T_2T_1T_4x_{n-1}^4}(t), F_{4\,x_n^4,T_3T_2T_1T_4x_n^4}(t),$$

$$F_{1\,T_4x_{n-1}^4,T_4x_n^4}(t), F_{2\,T_1T_4x_{n-1}^4,T_1T_4x_n^4}(t), F_{3\,T_2T_1T_3x_{n-1}^4,T_2T_1T_3x_n^4}(t)\} \tag{3.13}$$

$$F_{4x_n^4,x_{n+1}^4}(kt) \geq \min\{F_{4\,x_{n-1}^4,x_n^4}(t), F_{4x_{n-1}^4,x_n^4}(t), F_{4\,x_n^4,x_{n+1}^4}(t),\ F_{1\,x_{n-1}^1,x_n^1}(t), F_{2\,x_n^2,x_{n+1}^2}(t), F_{3x_n^3,x_{n+1}^3}(t)\}$$

$$F_{4x_n^4,x_{n+1}^4}(kt) \geq \min\{F_{1\,x_{n-1}^1,x_n^1}(t), F_{2x_{n-1}^2,x_n^2}(t), F_{3\,x_{n-1}^3,x_n^3}(t),\ F_{4\,x_{n-1}^4,x_n^4}(t)\}\ .$$

Using inequality (3.7), we get

$$F_{1x_n^1,x_{n+1}^1}(t) = F_{1\,T_4T_3T_2T_1x_{n-1}^1,\,T_4T_3T_2T_1x_n^1}(t)$$

$$\geq \min\{F_{1\,x_{n-1}^1,x_n^1}(t), F_{1x_{n-1}^1,T_4T_3T_2T_1x_{n-1}^1}(t), F_{1\,x_n^1,T_4T_3T_2T_1x_n^1}(t),$$

$$F_{2\,T_1x_{n-1}^1,T_1x_n^1}(t), F_{3\,T_2T_1x_{n-1}^1,T_2T_1x_n^1}(t), F_{4\,T_3T_2T_1x_{n-1}^1,T_3T_2T_1x_n^1}(t)\} \tag{3.14}$$

$$F_{1x_n^1,x_{n+1}^1}(kt) \geq \min\{F_{1\,x_{n-1}^1,x_n^1}(t), F_{1x_{n-1}^1,x_n^1}(t), F_{1\,x_n^1,x_{n+1}^1}(t),\ F_{2\,x_n^2,x_{n+1}^2}(t), F_{3\,x_n^3,x_{n+1}^3}(t), F_{4x_n^4,x_{n+1}^4}(t)\}$$

$$F_{1x_n^1,x_{n+1}^1}(kt) \geq \min\{F_{1\,x_{n-1}^1,x_n^1}(t), F_{2x_{n-1}^2,x_n^2}(t), F_{3\,x_{n-1}^3,x_n^3}(t),\ F_{4\,x_{n-1}^4,x_n^4}(t)\}\ .$$

By induction on using inequalities (3.11), (3.12), (3.13) and (3.14), we get

$$F_{1x_n^1,x_{n+1}^1}(k^{n-1}t) \geq \min\{F_{1\,x_1^1,x_2^1}(t), F_{2x_1^2,x_2^2}(t), F_{3\,x_1^3,x_2^3}(t),\ F_{4\,x_1^4,x_1^4}(t)\}$$

$$F_{2x_n^2,x_{n+1}^2}(k^{n-1}t) \geq \min\{F_{1\,x_1^1,x_2^1}(t), F_{2x_1^2,x_2^2}(t), F_{3\,x_1^3,x_2^3}(t),\ F_{4\,x_1^4,x_2^4}(t)\}$$

$$F_{3x_n^3,x_{n+1}^3}(k^{n-1}t) \geq \min\{F_{1\,x_1^1,x_2^1}(t), F_{2x_1^2,x_2^2}(t), F_{3\,x_1^3,x_2^3}(t), F_{4\,x_1^4,x_2^4}(t)\}$$

$$F_{4x_n^4,x_{n+1}^4}(k^{n-1}t) \geq \min\{F_{1\,x_1^1,x_2^1}(t), F_{2x_1^2,x_2^2}(t), F_{3\,x_1^3,x_2^3}(t), F_{4\,x_1^4,x_2^4}(t)\}\ .$$

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Since $k < 1$, it follows that $\{x_n^1\}$, $\{x_n^2\}$, $\{x_n^3\}$ and $\{x_n^4\}$ are Cauchy sequences with limit α_1, α_2, α_3 and α_4 in X_1, X_2, X_3 and X_4 respectively. Since T_1, T_2, T_3 and T_4 are continuous, we have

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} T_1 x_n^1 = T_1\alpha_1 = \alpha_2,$$

$$\lim_{n\to\infty} x_n^3 = \lim_{n\to\infty} T_2 x_n^2 = T_2\alpha_2 = \alpha_3,$$

$$\lim_{n\to\infty} x_n^4 = \lim_{n\to\infty} T_3 x_n^3 = T_3\alpha_3 = \alpha_4.$$

Using inequality (3.7) again, we have

$$F_{1\,T_4T_3T_2T_1\alpha_1, x_n^1}(kt) = F_{1\,T_4T_3T_2T_1\alpha_1,\, T_4T_3T_2T_1x_{n-1}^1}(t)$$

$$\geq \min\{F_{1\,\alpha_1, x_{n-1}^1}(t), F_{1\,\alpha_1, T_4T_3T_2T_1\alpha_1}(t), F_{1\,x_{n-1}^1, T_4T_3T_2T_1x_{n-1}^1}(t),\ F_{2\,T_1\alpha_1, T_1x_{n-1}^1}(t), F_{3\,T_2T_1\alpha_1, T_2T_1x_{n-1}^1}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1x_{n-1}^1}(t)\}.$$

Since T_1, T_2, T_3 are continuous, it follows on letting $n \to \infty$ that

$$F_{1\,T_4T_3T_2T_1\alpha_1, \alpha_1}(kt) = \min\{F_{1\,\alpha_1,\, T_4T_3T_2T_1\alpha_1}(t)\}.$$

Thus, we have $T_4T_3T_2T_1\alpha_1 = \alpha_1$, since $k < 1$ and α_1 is a fixed point of $T_4T_3T_2T_1$.

$$T_1T_4T_3T_2\alpha_2 = T_1T_4T_3T_2T_1\alpha_1 = T_1\alpha_1 = \alpha_2\ ,$$

$$T_2T_1T_4T_3\alpha_3 = T_2T_1T_4T_3T_2\alpha_2 = T_2\alpha_2 = \alpha_3 \text{ and}$$

$$T_3T_2T_1T_4\alpha_4 = T_3T_2T_1T_4T_3\alpha_3 = T_3\alpha_3 = \alpha_4.$$

Hence, α_2, α_3 and α_4 are the fixed points of $T_1T_4T_3T_2$, $T_2T_1T_4T_3$ and $T_3T_2T_1T_4$ respectively.

Uniqueness :

Suppose that $T_4T_3T_2T_1$ has another fixed point α_1.

Then using inequality (3.7), we have

$$F_{1\,\alpha_1, \alpha_1'}(kt) = F_{1\,T_4T_3T_2T_1\alpha_1,\, T_4T_3T_2T_1\alpha_1'}(t)$$

$$\geq \min\{F_{1\,\alpha_1, \alpha_1'}(t), F_{1\,\alpha_1, T_4T_3T_2T_1\alpha_1}(t), F_{1\,\alpha_1', T_4T_3T_2T_1\alpha_1'}(t),\ \ F_{2\,T_1\alpha_1, T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1, T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1'}(t)\} \quad (3.15)$$

$$F_{1\,\alpha_1, \alpha_1'}(kt) \geq \min\{F_{1\,\alpha_1, \alpha_1'}(t), F_{1\,\alpha_1, \alpha_1}(t), F_{1\,\alpha_1', \alpha_1'}(t),\ \ F_{1\,T_1\alpha_1, T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1, T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1'}(t)\}$$

Using inequality (3.8), we have

$$F_{2\,T_1\alpha_1, T_1\alpha_1'}(kt) = F_{2\,T_1T_4T_3T_2T_1\alpha_1,\, T_1T_4T_3T_2T_1\alpha_1'}(t)$$

$$\geq \min\{F_{2\,T_1\alpha_1, T_1\alpha_1'}(t), F_{2\,T_1\alpha_1, T_1T_4T_3T_2T_1\alpha_1}(t), F_{2\,T_1\alpha_1', T_1T_4T_3T_2T_1\alpha_1'}(t),$$

$$F_{3\,T_2T_1\alpha_1, T_1T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1'}(t), F_{4\,T_4T_3T_2T_1\alpha_1, T_4T_3T_2T_1\alpha_1'}(t)\}$$

$$F_{2\,T_1\alpha_1, T_1\alpha_1'}(kt) \geq \min\{F_{2\,T_1\alpha_1, T_1\alpha_1'}(t), F_{2\,T_1\alpha_1, T_1\alpha_1'}(t), F_{1\,T_1\alpha_1', T_1\alpha_1'}(t),$$

$$F_{3\,T_2T_1\alpha_1, T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1'}(t), F_{1\,T_4T_3T_2T_1\alpha_1, T_4T_3T_2T_1\alpha_1'}(t)\}$$

$$F_{2\,T_1\alpha_1, T_1\alpha_1'}(kt) \geq \min\{F_{2\,T_1\alpha_1, T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1, T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1'}(t),\ F_{1\,\alpha_1, \alpha_1'}(t)\}$$

$$F_{2\,T_1\alpha_1, T_1\alpha_1'}(kt) \geq \min\{F_{3\,T_2T_1\alpha_1, T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1'}(t), F_{1\,\alpha_1, \alpha_1'}(t)\}.$$

Now we write

$$F_{2\,T_1\alpha_1,T_1\alpha_1'}(kt) \geq \min\{F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t), F_{2\,T_1\alpha_1,T_1\alpha_1'}(t),$$

$$F_{3\,T_1\alpha_1,T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t), F_{1\,\alpha_1,\alpha_1'}(t)\} \quad (3.16)$$

$$F_{2\,T_1\alpha_1,T_1\alpha_1'}(kt) \geq \min\{F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t)\}.$$

Similarly, on using inequality (3.9), we have

$$F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(kt) \geq \min\{F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t), F_{2\,T_2T_1\alpha_1,T_2T_1T_4T_3T_2T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1',T_2T_1T_4T_3T_2T_1\alpha_1'}(t),$$

$$F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t), F_{1\,T_4T_3T_2T_1\alpha_1,T_4T_3T_2T_1\alpha_1'}(t), F_{2\,T_1T_4T_3T_2T_1\alpha_1,T_1T_4T_3T_2T_1\alpha_1'}(t)\}$$

$$F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(kt) \geq \min\{F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1',T_2T_1\alpha_1'}(t),$$

$$F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t), F_{1\,\alpha_1,\alpha_1'}(t), F_{2\,T_1\alpha_1,T_1\alpha_1'}(t)\} \quad (3.17)$$

$$F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(kt) \geq \min\{F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t), F_{1\,\alpha_1,\alpha_1'}(t), F_{2\,T_1\alpha_1,T_1\alpha_1'}(t)\}$$

$$F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(kt) \geq \min\{F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t), F_{2\,T_1\alpha_1,T_1\alpha_1'}(t), F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t), F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t)\}.$$

Using inequality (3.16) and (3.17), we have

$$F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(kt) \geq \min\{F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t)\}. \quad (3.18)$$

Similarly, on using inequalities (3.10), (3.16) and (3.18), we have

$$F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(kt) \geq F_{1\,\alpha_1,\alpha_1'}(t). \quad (3.19)$$

Using inequalities (3.15), (3.16), (3.18) and (3.19), we have

$$F_{1\,\alpha_1,\alpha_1'}(kt) \geq F_{2\,T_1\alpha_1,T_1\alpha_1'}(t)$$

$$F_{1\,\alpha_1,\alpha_1'}(k^2t) \geq F_{3\,T_2T_1\alpha_1,T_2T_1\alpha_1'}(t)$$

$$F_{1\,\alpha_1,\alpha_1'}(k^3t) \geq F_{4\,T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1'}(t)$$

$$F_{1\,\alpha_1,\alpha_1'}(k^4t) \geq F_{1\,\alpha_1,\alpha_1'}(t).$$

Now, we have

$$F_{1\,\alpha_1,\alpha_1'}(k^4t) \geq F_{1\,\alpha_1,\alpha_1'}(t).$$

Since $0 \leq k < 1$, we have

$$F_{1\,\alpha_1,\alpha_1'}(t) = 0$$

$$\Rightarrow \qquad \alpha_1 = \alpha_1'$$

proving the uniqueness of α_1.

We can similarly prove that $T_1T_4T_3T_2$ has a unique fixed point α_2 in X_2, $T_2T_1T_4T_3$ has a unique fixed point α_3 in X_3 and $T_3T_2T_1T_4$ has a unique fixed point α_4 in X_4.

IV. CONCLUSION

In this paper, we established two related fixed point theorems for four mappings in four Menger spaces which generalizes and extends the result of Gupta [4] in four metric spaces.

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