Common Fixed Point Theorems in Menger Spaces

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ABSTRACT

The concept of Menger space has been introduced recently as a generalization of metric space. The aim of this paper is to use the concept of occasionally weakly compatible mappings and semi-compatible mappings in Menger space and prove a common fixed point theorem.

Keywords: *Common fixed point, Menger space, compatible maps, semi compatible maps, reciprocal continuous*

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1. INTRODUCTION

In 1942, Menger [8] has introduced the theory of probabilistic metric space by introducing probabilistic notion into geometry. The study of contraction mapping theorem was initiated by Sehgal [12] in 1966 in PM- space. Altun and Turkoglu [2] proved two common fixed point theorems on complete PM- space with an implicit relation. Schweizer and Sklar [11] played major role in development of fixed point theory in PM - space. In 1972, Sehgal and Bharucha- Reid [13] initiated the study of contraction mappings in the development of fixed point theorems. Singh et. al. [15] introduced the concept of weakly commuting mapping in PM- space. Kumar and Chugh [7] established some common fixed point theorem using the idea of reciprocal continuous of mappings.

Recently Al- Thagafi and Shahzad [1] weakened the notion of weakly compatible maps by introducing owc maps. Bouhadjera and Godet-Thobie [3] introduced two new notions subsequential continuity and subcompability which are weaker than reciprocal continuity and compatibility respectively. Using compatibility of type (A), Jain et. al. [4] proved an interesting result.

2. PRELIMINARIES

Definition 2.1.[9] A mapping **F** : R $\rightarrow$ R^+ is called a *distribution* if it is non-decreasing left continuous with

inf { **F** (t) | t $\in$ R } = 0 and sup { **F** (t) | t $\in$ R} = 1.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

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$$H(t) = \begin{cases} 0 & , \quad t \leq 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [9] A mapping t :[0, 1] × [0, 1] □ [0, 1] is called a *t-norm* if it satisfies the following conditions :

(t-1) t(a, 1) = a, t(0, 0) = 0 ;

(t-2) t(a, b) = t(b, a) ;

(t-3) t(c, d) □ t(a, b) ; for c □ a, d □ b,

(t-4) t(t(a, b), c) = t(a, t(b, c)) for all a, b, c, d □[0, 1].

Definition 2.3. [9] A *probabilistic metric space (PM-space)* is an ordered pair (X, **F**) consisting of a non-empty set X and a function **F** : X × X □ L, where L is the collection of all distribution functions and the value of **F** at (u, v) □ X × X is represented by $F_{u, v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all x > 0, if and only if u = v;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,

for all u,v,w □ X and x, y > 0.

Definition 2.4. [9] A *Menger space* is a triplet (X, **F**, t) where (X, **F**) is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y)$ □ t {$F_{u, v}(x)$, $F_{v, w}(y)$ }, for all u, v, w□ X, x, y □ 0.

Definition 2.5. [9] A sequence {x_n} in a Menger space (X, **F**, t) is said to be *convergent* and *converges to a point* x in X if and only if for each □ > 0 and □ > 0, there is an integer M(□, □) such that

$F_{x_n, x}$(□) > 1 - □ for all n □ M(□, □).

Further the sequence {x_n} is said to be *Cauchy sequence* if for □ > 0 and □ > 0, there is an integer M(□, □) such that

F_{x_n, x_m}(□) > 1- □ for all m, n □ M(□, □).

A Menger PM-space (X, **F**, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

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A complete metric space can be treated as a complete Menger space in the following way :

Proposition 2.1. [9] If (X, d) is a metric space then the metric d induces mappings **F** : $X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

$$H(k) = 0, \quad \text{for } k \le 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a,b) = \min\{a, b\}$. Then (X, **F**, t) is a Menger space. It is complete if (X, d) is complete.

The space (X, **F**, t) so obtained is called the *induced Menger space.*

Definition 2.6. [5] Self mappings A and S of a Menger space (X, **F**, t) are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [9] Self mappings A and S of a Menger space (X, **F**, t) are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [10] Self maps S and T of a Menger space (X, **F**, t) are said to be *semi-compatible* if $F_{STx_n, Tu}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.9. [1] Self maps A and S of a Menger space (X, **F**, t) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Definition 2.10. [6] Two self maps P and S of a Menger space (X, **F**, t) are said to be reciprocally continuous if $PSx_n \to Pz$ and $SPx_n \to Sz$, whenever $\{x_n\}$ is a sequence in X such that $Px_n, Sx_n \to z$, for some z in X as $n \to \infty$.

Lemma 2.1. [14] Let (X,F,*) be a Menger space with continuous t- norm * , if there exists a constant $h \in (0,1)$ such that $F_{x,y}(ht) \ge F_{x,y}(t)$, for all $x,y \in X$, and $t > 0$ then $x = y$.

Example 1.1. Let M = [4, 40] and d be usual metric on M. Define mappings P, S : $M \to M$ by

$$Pv = \begin{cases} 4, & \text{if} \quad v = 4 \\ 5, & \text{if} \quad v > 4 \end{cases} \quad \text{and} \quad Sv = \begin{cases} 4, & \text{if} \quad v = 4 \\ 20, & \text{if} \quad v > 4 \end{cases}.$$

It is noted that P and S are reciprocally continuous mappings but they are not continuous.

Lemma 2.2. [14] Let $\{x_n\}$ be a sequence in a Menger space (X,**F**, t), where t is continuous and satisfies $t(x,y) \geq x$, for all $x \in [0,1]$. If there exists a constant $k \in (0,1)$ such that

$F_{u_n,u_{n+1}}(kx) \geq F_{u_{n-1},u_n}(x)$, n = 1, 2, 3, … then $\{x_n\}$ is

a Cauchy sequence in X.

3. MAIN RESULT

Theorem 3.1. Let P, Q, S and T be self mappings on a complete Menger space (X, **F**, t) with continuous t-norm $t(c, c) \geq c$, for some $c \in [0,1]$ satisfying :

(3.1) $P(X) \subseteq T(X)$, $Q(X) \subseteq S(X)$,

(3.2) (Q, T) is occasionally weak compatible,

(3.3) For all $x,y \in X$, and $h > 1$,

$$F_{Px,Qy}(hx) \geq \text{Min}[F_{Sx,Ty}(x), \{F_{Sx,Px}(x).\,F_{Qy,Ty}(x)\}, F_{Px,Sx}(x)]$$

If (P, S) is semi compatible pairs of reciprocal continuous maps then P, Q, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$, be any arbitrary point. Then we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Px_{2n+1} = Tx_{2n}$, and $y_{2n+1} = Qx_{2n+2} = Sx_{2n+1}$, for n = 0,1,2,…

First, we will prove that $\{y_n\}$ is a Cauchy sequence in X.

Now, by inequality (3.3), we have

$$F_{y_{2n+1},y_{2n+2}}(hx) \geq \text{Min}[F_{Sx_{2n+1},Tx_{2n+2}}(x), \{F_{Sx_{2n+1},Px_{2n+1}}(x).\,F_{Qx_{2n+2},Tx_{2n+2}}(x)\}, F_{Px_{2n+1},Sx_{2n+1}}(x)]$$

$$\geq \text{Min}[F_{y_{2n+1},y_{2n+2}}(x), \{F_{y_{2n+1},y_{2n}}(x).F_{y_{2n+1},y_{2n+2}}(x)\}, F_{y_{2n+1},y_{2n+2}}(x)]$$

$$F_{y_{2n+1},y_{2n+2}}(hx) \geq F_{y_{2n},y_{2n+1}}(x).$$

Similarly , we get

$$F_{y_{2n+2},y_{2n+3}}(hx) \geq F_{y_{2n+1},y_{2n+2}}(x).$$

In general, we have

$$F_{y_{n+1},y_n}(hx) \geq F_{y_n,y_{n-1}}(x).$$

Then by Lemma 2.2 , $\{y_n\}$ is a Cauchy sequence and it converges to some point z in X.

Hence the subsequences convergent as follows :

$\{Px_{2n}\} \to z$, $\{Sx_{2n}\} \to z$, $\{Qx_{2n+1}\} \to z$ and $\{Tx_{2n+1}\} \to z$.

Now, since P and S are reciprocal continuous and semi- compatible then we have $\lim_{n\to\infty} PSx_{2n} = Pz$,

$\lim_{n\to\infty} SPx_{2n} = Sz$, and $\lim_{n\to\infty} M(PSx_{2n}, Sz,t) = 1$. Therefore we get Pz = Sz .

Now we will show that Pz = z .

By inequality (3.3) , putting $x = z$, $y = x_{2n+1}$, we get

$$F_{Pz,Qx_{2n+1}}(hx) \geq \text{Min}[F_{Sz,Tx_{2n+1}}(x), \{F_{Sz,Pz}(x). F_{Qx_{2n+1},Tx_{2n+1}}(x)\}, F_{Pz,Sz}(x)].$$

Taking limit $n\to\infty$, we get

$$F_{Pz,z} (hx) \geq \text{Min} [F_{sz,z}(x), \{F_{Sz,Pz} (x) . F_{z,z}(x) \}, F_{Pz,Sz}(x)]$$

Since Pz =Sz , then we get

$F_{Pz,z} (hx) \geq \text{Min} [F_{Pz,z} (x) , \{ F_{Pz,Pz} (x) . F_{z,z} (x) \}, F_{Pz,Pz} (x)]$

$F_{Pz,z}(hx) \geq F_{Pz,z}(x)$,

then by Lemma 2.1, then we get z = Pz.

Since, Pz = Sz , combining both we get z = Pz = Sz.

Now , $P(X) \subseteq T(X)$, therefore there exists a point $u \in X$ such that z = Pz = Tu.

Putting $x= x_{2n}$, y = u in inequality (3.3) , we get

$$F_{Px_{2n},Qu} (hx) \geq \text{Min}[F_{Sx_{2n},Tu} (x), \{F_{Sx_{2n},Px_{2n}}(x). F_{Qu,Tu}(x)\}, F_{Px_{2n},Sx_{2n}}(x)]$$

Letting $n\to\infty$, we get

$$F_{z, Qu} (hx) \geq \text{Min} [F_{z, Tu} (x) , \{ F_{z,z}(x) . F_{Qu,z} (x)\}, F_{z,z} (x)]$$

$$F_{z,Qu} (hx) \geq F_{z,Tu} (x)$$

Then, by Lemma 2.1, we get Qu = Tu .

Since z = Pz = Tu and we proved that Qu = Tu , combining both we get

z = Qu = Tu.

Occasionally weak compatibility of (Q, T) gives TQu = QTu i.e. Qz = Tz .

Now , we will prove that Qz = Pz .

Again assuming $Qz \neq Pz$.

By inequality (3.3), putting x = z , y = z , we get

$F_{Pz,Qz}$ (hx) $\geq$ Min [$F_{Sz,Tz}$ (x) , { $F_{Sz,Pz}$ (x) . $F_{Qz,Tz}$ (x) } , $F_{Pz,Sz}$ (x)]

$F_{Pz,Qz}$ (hx) $\geq$ Min [$F_{Pz,Qz}$ (x) , { $F_{Pz,Pz}$ (x) . $F_{Qz,Qz}$ (x) } , $F_{Pz,Pz}$ (x)]

$F_{Pz,Qz}$ (hx) $\geq$ $F_{Pz,Qz}$ (x),

which is a contradiction , thus we get Pz = Qz.

Since Pz = Sz = z , and Qz = Tz.

Hence finally we get

z = Pz = Qz = Sz = Tz. i.e. z is a common fixed point of P, Q, S and T.

Uniqueness: Let w be another common fixed point of P, Q, S and T, then w = Pw = Qw = Sw = Tw .

Putting x = z and y = w , in inequality (3.3) , we get

$F_{Pz, Qw}$ (hx) $\geq$ Min [$F_{Sz,Tw}$ (x), {$F_{Sz,Pz}$ (x) . $F_{Qw,Tw}$ (x)} , $F_{Pz,Sz}$ (x)]

$F_{z,w}$ (hx) $\geq$ Min [$F_{z,w}$ (x) , { $F_{z,z}$ (x) . $F_{w,w}$ (x) } , $F_{z,z}$ (x)]

$F_{z,w}$ (hx) $\geq$ $F_{z,w}$ (x)

Hence , from Lemma 2.1 , we get z = w .

Therefore z is a unique common fixed point of P,Q,S and T .

By setting P = Q in theorem 3.1, we have the following corollary -

Corollary 3.2. Let P, S and T be self maps of a complete Menger space (X,**F**,t) , where t is continuous t-norm, satisfying following conditions :

1. The pair (P,T) is occasionally weak compatible,
2. For all x,y $\in$ X and h > 1,

   $F_{Px,Py}$ (hx) $\geq$ Min [$F_{Sx,Ty}$(x), { $F_{Sx,Px}$ (x) . $F_{Py,Ty}$ (x) }, $F_{Px,Sx}$ (x)]

If (P, S) is semi compatible pairs of reciprocally continuous maps Then , P,S and T have a unique common fixed point in X.

On taking P = Q and S = T , we get another corollary -

Corollary 3.3. Let P and S be self maps of a complete Menger space (X, **F**, t) , where t is continuous t-norm, satisfying following conditions :

1. For all x,y ∈ X and h > 1,

$$F_{Px,Py}(hx) \geq \text{Min}\left[F_{Sx,Sy}(x), \{ F_{Sx,Px}(x) . F_{Py,Sy}(x) \}, F_{Px,Sx}(x) \right]$$

If (P,S) is semi compatible pairs of reciprocally continuous maps and occasionally weak compatible. Then, P and S have a unique common fixed point in X.

REFERENCES

1. Al-Thagafi, M.A. and Shahzad, N., Generalized I- nonexpensive selfmaps and invariant approximations, *Acta Math. Sinica*, 24(5) (2008) 867-876.
2. Altun, I. and Turkoglu, D., Some fixed point theorems on fuzzy metric spaces with implicit relations, *Commun. Korean Math. Soc.*, 23(1) (2008) 111-124.
3. Bouhadjera, H. and Godet–Thobie, C., Common fixed point theorems for pairs of sub-compatible maps, arXiv : 0906 .3159v2, (2009).
4. Jain, A. and Singh, B., Common fixed point theorems in Menger space through compatible maps of type (A), *Chh. J. Sci.Tech.*, 2 (2005) 1-12.
5. Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, *Indian Pure Appl. Math.*, 29(3) (1998) 227-238.
6. Kumar, S. and Pant, B.D., A common fixed point theorem in probabilistic metric space using implicit relatios, *Filomat*, 22(2) (2008) 43-52.
7. Kumar, S. and Chugh, R., Common fixed point theorems using minimal commutativity and reciprocal continuity conditions in metric spaces, *Sci. Math. Japan*, 56 (2002) 269-275.
8. Menger, K., Statistical metrics, *Poc. Nat. Acad. Sci.*, 28 (1942) 535-537.
9. Mishra, S.N., Common fixed points of compatible mappings in PM-space, *Math. Japon.*, 36(2) (1991) 283-289.
10. Pant, B.D. and Chouhan, S., Common fixed point theorems for semi-compatibility maps using implicit relation, *Int. J. of Math. Analysis*, 3(28) (2009) 1389-1398.
11. Schweizer, B.and Sklar, A., Probabilistic Metric Spaces, North Holland (Amsterdam 1983).
12. Sehgal, V.M., Some fixed point theorems in functions analysis and probability, Ph.D. dissertation, Wayne State Univ. Michigan (1966).
13. Sehgal, V.M. and Bharucha-Reid, A.T., Fixed points of contraction mappings on probabilistic metric spaces, *Math. Systems Theory*, 6(1972) 97-102.
14. Singh, S.L. and Pant, B.D., Common fixed point theorems in probabilistic metric space and extension to uniform spaces, *Honam. Math. J.*, (1984) 1-12.
15. Singh, S.L., Pant, B.D. and Talwar, R., Fixed points of weakly commuting mappings on Menger spaces, *Jnanabha*, 23 (1993) 115-122.