International Journal of Advanced Technology in Engineering and Science
Vol. No.7, Issue No. 02, February 2019
www.ijates.com
ijates
ISSN 2348 - 7550

COMMON FIXED POINTS OF COMPATIBLE MAPS IN PROBABILISTIC 2-METRIC SPACE

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ABSTRACT

In this paper, the concept of compatibility of type (A) in probabilistic 2-metric space has been applied to prove a common fixed point theorem for four mappings. Our result generalizes and extends the result of Singh and Chauhan [9].

Keywords : *Probabilistic 2-metric space, Menger space, common fixed point, compatible maps, compatible maps of type (A).*

AMS Subject Classification : *Primary 47H10, Secondary 54H25.*

1. Introduction.

The concept of probabilistic metric space was initially investigated by Menger [6] as a new way to represent vagueness in everyday life. Subsequently, it was developed extensively by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined probabilistic metric (PM) space in various ways.

In this paper, we deal with the PM space defined by Schweizer and Sklar [8] and modified by Mishra [7], Singh and Jain [10]. Jungck [3,4] gave the more generalized concept compatibility than commutativity and weak commutativity in metric space and proved common fixed point theorems. Jungck et. al. [5] proposed the concept of compatible maps of type (A) in Menger space and gave some fixed point theorems. Recently, using the concept of compatible mappings of type (A), Jain et. al. [1, 2] proved some interesting fixed point theorems in Menger space.

In this paper a fixed point theorem for four self maps has been proved using the concept of compatible maps of type (A) which generalizes and extends the result of Singh and Chauhan [9].

2. Preliminaries.

Throughout this paper, we use symbols and basic definitions of Jain and Singh [1].

Definition 2.1. [7] A mapping $F : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with

$$\inf \{ F(t) \mid t \in R \} = 0 \quad \text{and} \quad \sup \{ F(t) \mid t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

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ISSN 2348 - 7550

$$H(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [11] A *probabilistic 2-metric space (2-PM space)* is an ordered pair (X, F) where X is an abstract set and F is a function defined on $X \times X \times X$ into L, the collection of all distribution functions. The value of F at $(x, y, z) \in X \times X \times X$ is generally represented by $F_{x,y,z}$ or F(x, y, z). The distribution function F(x, y, z) satisfy the following conditions :

(1) F(x, y, z; 0) = 0,

(2) For all distinct x, y in X there exists a point z in X such that
F(x, y, w; t) < 1 for some t > 0.

(3) F(x, y, z; t) = 1 for all t > 0 if and only if at least two of the three points are equal.

(4) F(x, y, z; t) = F(x, z, y ; t) = F(y, z, x; t) (Symmetry)

(5) If $F(x, y, z; t_1) = F(x, z, y; t_2) = F(z, y, x; t_3) = 1$ then
$F(x, y, z, ; t_1 + t_2 + t_3) = 1$.

Definition 2.3. [11] The mapping $t : [0, 1] \times [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a *t-norm* if t satisfies the following conditions :

(1) t(x, 1, 1) = x , t(0, 0, 0) = 0 ;

(2) t(x, y, z) = t(x, z, y) = T(z, y, x) ;

(3) $t(x_1, y_1, z_1) \ge t(x_2, y_2, z_2)$ for $x_1 \ge x_2$, $y_1 \ge y_2$, $z_1 \ge z_2$;

(4) t(t(x, y, z), p, q) = t(x, t(y, z, p), q) = t(x, y, t(z, p, q)).

Definition 2.4. [11] A *Menger probabilistic 2-metric space* is a triplet (X, F, t) where (X, F) is a 2-PM space and t is a t-norm satisfying the following triangle inequality :

$$F(x, y, z; t_1 + t_2 + t_3) \ge y(F(x, y, p; t_1), F(x, p, z; t_2), F(p, y, z; t_3)$$

for all $x, y, z, p \in X$ and $t_1, t_2, t_3 \ge 0$.

Definition 2.5. [11] A sequence $\{x_n\}$ in a 2-Menger space (X, F, t) is said to *converge* to a point $x \in X$ if for each $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer $M(\varepsilon,\lambda)$ such that

$$F(x_n, x, a; \varepsilon) > 1 - \lambda, \quad \text{for all } a \in X \text{ and } n \ge M(\varepsilon, \lambda).$$

The sequence $\{x_n\}$ converges to x if and only if

$$F(x_n, x, a; t) = H(t) \quad \text{for all } a,$$

where H is the distribution function defined as above.

Definition 2.6. [11] A sequence $\{x_n\}$ in a 2-Menger space (X, F, t) is said to be *Cauchy* if, for each $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer $M(\varepsilon, \lambda)$ such that

$$F(x_n, x_m, a; \varepsilon) > 1 - \lambda, \text{ for all } a \in X \text{ and } n, m \ge M(\varepsilon, \lambda).$$

Lemma 2.1. [11] Let $\{x_n\}$ be a sequence in a 2-Menger space (X, **F**, t) where t is continuous and satisfies $t(x, x, x) \ge x$ for all $x \in (0, 1)$. If there exists a positive number h < 1 such that

$F(x_{n+1}, x_n, a; hu) \geq F(x_n, x_{n-1}, a; u)$, $n = 1, 2, 3, ...$

for all $a \in X$ and $u \geq 0$ then $\{x_n\}$ is a Cauchy sequence in X.

Definition 2.7. [10] Self mappings A and S of a Menger probabilistic 2-metric space (X, F, t) are said to be *compatible* if $F_{ASx_n, SAx_n, a}(x) \to 1$ for all $a \in X$, $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.7. [1] Self maps S and T of a Menger probabilistic 2-metric space (X, F, t) are said to be *compatible of type (A)* if $F_{STx_n, TTx_n, a}(x) \to 1$ and $F_{TSx_n, SSx_n, a}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

Proposition 2.2. [1] Let S and T be compatible maps type of (A), self maps of a Menger space X, let $Sx_n, Tx_n \to u$ for some u in X. Then

(a) $TSx_n \to Su$ if S is continuous.

(b) $STu = TSu$ and $Su = Tu$ if S and T are continuous.

Proposition 2.3. [1] If S and T are compatible self maps of a Menger space (X, F, t) where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Sx_n, Tx_n \to u$ for some u in X. Then $TSx_n \to u$ provided S is continuous.

Proposition 2.4. [1] If S and T are continuous and compatible self maps of a Menger space (X, F, t) where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$. Then S and T are compatible maps of type (A).

Proposition 2.5. [1] Let S and T be compatible maps of type (A), self maps of a Menger space (X, F, t), where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$. If one of S and T is continuous then S and T are compatible.

Proposition 2.6. [1] Let S and T be compatible maps of type (A), self maps of Menger space X and Su = Tu for some u in X then STu = TSu = SSu = TTu.

Lemma 2.1. [10] Let $\phi : [0, \infty) \to [0, \infty)$ be strictly increasing left continuous function such that $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = +\infty$ and $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$, for all $t > 0$.

Define function $\Psi : [0, \infty) \to [0, \infty)$ by

$\Psi(0) = 0$ and $\Psi(t) = \inf\{ s > 0 : f(s) > t \}$, $t > 0$. Then

(i) $\phi(t) < t$ for all $t > 0$,

(ii) $\phi(\Psi(t)) \leq t$ and $\psi(\phi(t)) = t$ for all $t \geq 0$,

(iii) $\Psi(t) \geq 1$ for all $t \geq 0$,

(iv) $\lim_{n\to\infty} \Psi^n(t) = +\infty$ for all $t > 0$.

Lemma 2.2. [10] $\Psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function, where Ψ is defined as in Lemma 2.1.

Lemma 2.3. [10] Let $\{y_n\}$ be a sequence in a Menger space (X, **F**, min) and the function Ψ defined as in Lemma 2.1 such that for all $x > 0$ and $n \in N$,

$$F_{y_n, y_{n+1}}(x) \geq F_{y_0, y_1}(\Psi^n(x)).$$

Then $\{y_n\}$ is a Cauchy sequence in X.

3. MAIN RESULT.

Theorem 3.1. Let A, B, S and T be self mappings of a complete probabilistic 2-metric space (X, F, *) with continuous t-norm * defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$, satisfying the following conditions:

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(3.2) One of A, B, S and T is continuous;

(3.3) (A, S) and (B, T) are pairs of compatible maps of type (A);

(3.4) $F_{Ap,Bq,a}(\phi(x)) \geq \min\{F_{Sp,Tq,a}(x), F_{Bq,Tq,a}(x), F_{Sp,Ap,a}(x), F_{Ap,Tq,a}(ax), F_{Bq, Sp,a}((2 - \alpha)x)\}$,

for all $p, q \in X$, $x > 0$, $\alpha \in (0, 2)$ where ϕ is defined as in Lemma 2.1.

Then A, B, S and T have a unique common fixed point in X.

Proof. In virtue of condition (3.1), we construct a sequence $\{y_n\}$ in X such that

$$y_{2n-1} = Ax_{2n-2} = Tx_{2n-1};$$
$$y_{2n} = Bx_{2n-1} = Sx_{2n}; \quad n = 1, 2, 3, \ldots .$$

Using (3.4) and Lemma 2.1, we have

$$\begin{aligned} F_{y_{2n+1},y_{2n+2},a}(x) &\geq F_{Ax_{2n}, Bx_{2n+1},a}(\phi(\Psi(x))) \\ &\geq \min\{F_{Sx_{2n}, Tx_{2n+1},a}(\Psi(x)), F_{Bx_{2n+1},Tx_{2n+1},a}(\Psi(x)), \\ &\qquad F_{Sx_{2n}, Ax_{2n},a}(\Psi(x)), F_{Ax_{2n}, Tx_{2n+1},a}(\alpha\Psi(x)), \\ &\qquad F_{Bx_{2n+1}, Sx_{2n},a}((2 - \alpha)\Psi(x))\}. \end{aligned}$$

Let $\beta \in (0,1)$ and put $\beta = 1 - \alpha$, we get

$$\begin{aligned} F_{y_{2n+1}, y_{2n+2},a}(x) \geq\ & \min\{F_{y_{2n}, y_{2n+1},a}(\Psi(x)),F_{y_{2n+2}, y_{2n+1},a}(\Psi(x)), \\ &\qquad F_{y_{2n}, y_{2n+1},a}(\Psi(x)), F_{y_{2n+1}, y_{2n+1},a}((1- \beta)\,\Psi(x)), \\ &\qquad F_{y_{2n+2}, y_{2n},a}((1+ \beta)\,\Psi(x))\} \\ \geq\ & \min\{F_{y_{2n}, y_{2n+1},a}(\Psi(x)), F_{y_{2n+1}, y_{2n+2},a}(\Psi(x)), \\ &\qquad \min\{F_{y_{2n}, y_{2n+1},a}(\Psi(x)), F_{y_{2n+1}, y_{2n+2},a}(\beta\Psi(x))\}. \end{aligned}$$

Making $\beta \to 1$,we get

International Journal of Advanced Technology in Engineering and Science
Vol. No.7, Issue No. 02, February 2019
www.ijates.com
ijates
ISSN 2348 - 7550

$$F_{y_{2n+1},\ y_{2n+2},\ a}(x) \ \geq\ \min\{F_{y_{2n},\ y_{2n+1},a}(\Psi(x)), F_{y_{2n+1},\ y_{2n+2},a}(\Psi(x))\}.$$

Similarly,

$$F_{y_{2n+2},\ y_{2n+3},\ a}(x) \ \geq\ \min\{F_{y_{2n+1},\ y_{2n+2},a}(\Psi(x)), F_{y_{2n+2},\ y_{2n+3},a}(\Psi(x))\}.$$

In general,

$$\begin{aligned} F_{y_n,\ y_{n+1},a}(x) &\geq \min\{F_{y_{n-1},\ y_n,a}(\Psi(x)), F_{y_n,\ y_{n+1},a}(\Psi(x))\} \\ &\geq \min\{F_{y_{n-1},\ y_n,a}(\Psi(x)), F_{y_n,\ y_{n+1},a}(\Psi^2(x))\} \\ &\geq \quad \dots\ \dots\ \dots \\ &\geq \min\ \{F_{y_{n-1},\ y_n,a}(\Psi(x)), F_{y_n,\ y_{n+1},a}(\Psi^r(x))\}. \end{aligned}$$

By Lemma 2.1,

$$\Psi^2(x) \geq \ \Psi(x) \quad \text{and} \quad F_{x,y,a}(\Psi^2(x)) \geq F_{x,y\ ,a}(\Psi(x)).$$

Letting $r \to \infty$, we have $F_{y_n,\ y_{n+1},a}(\Psi^r(x)) \to 1$, yields

$$F_{y_n,\ y_{n+1},a}(x) \ \geq\ F_{y_{n-1},\ y_n,a}(\Psi(x)) \ \geq\ \dots\ \geq\ F_{y_0,\ y_1,a}(\Psi^n(x)).$$

Therefore by Lemma 2.3, $\{y_n\}$ is a Cauchy sequence in X.

By completeness of X, $\{y_n\}$ and also its subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ converge to some z in X.

Suppose that T is continuous then TTx_{2n-1}, $TBx_{2n-1} \to Tz$. Since B, T are compatible of type (A), then by Proposition 2.2, $BTx_{2n-1} \to Tz$.

Using (3.4), we have

$$\begin{aligned} F_{Ax_{2n-1},\ BTx_{2n-1},a}(x) \ \geq \min\ \{&F_{Sx_{2n-2},\ T^2x_{2n-1},a}(\Psi(x)), F_{BTx_{2n-1},T^2x_{2n-1},a}(\Psi(x)), \\ &F_{Sx_{2n-2},\ Ax_{2n-2},\ a}(\Psi(x)), F_{Ax_{2n-2},\ T^2x_{2n-1},a}(\alpha\Psi(x)), \\ &F_{BTx_{2n-1},\ Sx_{2n-2},\ a}((2 - \alpha)\Psi(x))\}. \end{aligned}$$

Letting $n \to \infty$ and making $\alpha \to 1$, we obtain

$$F_{z,\ Tz,\ a}(x) \ \geq\ F_{z,\ Tz,\ a}(\Psi(x)).$$

Inductively, we have

$$\begin{aligned} F_{z,\ Tz,\ a}(x) \ &\geq\ F_{z,\ Tz,a}(\Psi(x)) \ \geq\ F_{z,\ Tz,\ a}(\Psi^2(x)) \\ &\geq\ \dots\ \geq\ F_{z,\ Tz,\ a}(\Psi^r(x)) \ \to 1 \quad \text{as } r \to \infty, \quad \text{yields} \end{aligned}$$

$$Tz \ =\ z.$$

Similarly,

$$Bz = \ z.$$

Since $B(X) \subseteq S(X)$, there exists point u in X such that

$$z = Bz = Su.$$

Again using (3.4), we have

$$F_{Au, z, a}(x) = F_{Au, Bz, a}(x)$$
$$\geq \min \{F_{z, z, a}(y(x)), F_{z, z, a}(y(x)), F_{z, Au, a}(y(x)),$$
$$F_{Au, z, a}(aY(x)), F_{z, z, a}((2 - \alpha)Y(x))\}.$$

Making $\alpha \to 1$, we get

$$F_{Au, z, a}(x) \geq F_{Au, z, a}(\Psi(x)), \quad \text{yields}$$
$$Au = z.$$

Since A, S are compatible of type (A) and z = Au = Su, by Proposition 2.6,

$$Az = ASu = SAu = Sz.$$

Using (3.4), we have

$$F_{Az, z, a}(x) = F_{Az, Bz, a}(x)$$
$$\geq \min \{F_{Az, z, a}(\Psi(x)), F_{z, z, a}(\Psi(x)), F_{Az, Az, a}(\Psi(x)),$$
$$F_{Az, z, a}(\alpha \Psi(x)), F_{z, Az, a}((2 - \alpha)\Psi(x))\}.$$

Making $\alpha \to 1$, we get

$$F_{Az, z, a}(x) \geq F_{Az, z, a}(Y(x)), \quad \text{yields}$$
$$Az = z.$$

Therefore,

$$Az = Bz = Sz = Tz = z.$$

Now for uniqueness of z, let z' be another common fixed point of A, B, S and T then from (3.4), we have

$$F_{z, z', a}(x) = F_{Az, Bz', a}(x)$$
$$\geq \min \{F_{z, z', a}(\Psi(x)), F_{z', z', a}(\Psi(x)), F_{z, z, a}(\Psi(x)),$$
$$F_{z, z', a}(\alpha\Psi(x)), F_{z', z, a}((2 - \alpha)\Psi(x))\}.$$

Making $\alpha \to 1$, we get

$$F_{z, z', a}(x) \geq F_{z, z', a}(\Psi(x)), \quad \text{yields}$$
$$z = z'.$$

This completes the proof.

As a corollary of theorem 3.1, we obtain the following result.

Corollary 3.1. Let A, B, S and T be self mappings of a complete Menger space (X, F, min) satisfying (3.1), (3.4) and

(3.5) S and T are continuous.

(3.6) (A, S) and (B, T) are pairs of compatible maps,

Then the conclusion of theorem 2.2.1 holds.

Proof. From (3.5) if S and T are continuous they by Proposition 2.4, (3.6) implies (3.3). Hence the proof of the corollary.

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