International Journal of Advanced Technology in Engineering and Science
Vol. No. 11, Issue No. 05, May 2023
www.ijates.com
ijates
ISSN 2348 - 7550

A Mathematical Study on Some Special Aspects of a Peculiar Ecological Model -Global Stability & Diffusion Analysis

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Abstract

This research devotes a lot of attention to the ecosystem of the three species, and it does so by developing a model that includes Ammensal- *EnemyHost- Commensal (AEC) species. All of which have varying rates that are proportionate to their respective population levels. The main focus of this investigation is an examination of the state of global stability. The research is carried out with an emphasis in view of diffusion analysis.*

Key words: Stability, Global Stability, Routh-Hurwitz Criterion, Diffusion Analysis.
Mathematics Subject Classification34D10,34D20,34D23,35C10, 92D25,92D40.

1. Introduction:

Ecology leads to a deeper comprehension of the fundamental acts as they are today and how they can continue to grow and endure in the future. Ecology improves our environment, which is crucial for human prosperity and well-being. It provides fresh perspectives on the relationship between nature and living things. Therefore, it is crucial for each person's development along with the preservation of natural resources. Many mathematical models of ecology have been examined for their local and global stability in terms of multicultural features by N.Ch. Pattabhi Ramacharyulu & K.V.L.N.Acharyulu [2–9], Numerous scholars [10–17] and mathematicians [1,18] have contributed significant viewpoints and useful ideas.

In the topic of mathematical ecology, there are fascinating approaches for analyzing the behaviour of different models in ecology. Local stability demonstrates stability over modest shocks, whereas global stability reflects a system's high resistance to changes in internal dynamics and external effects.

Therefore, solving mathematical problems, translating real-world issues into mathematical terms, and analyzing those terms in the context of the real world are all fundamental components of mathematical modeling. Rarely a real-world issue can be translated into a mathematical issue. Sometimes it is impossible to answer the ensuing mathematical conundrum. Therefore, in order to trace the correct behavior and draw the appropriate conclusions, it would be essential to "simplify" or "idealize" or "approximate" the original problem by treating it as a replica that is also meaningful and mathematically translated.

1.1 Notations Adopted for three species:

Within this ecosystem, the population strength of Ammesnal species is denoted by the value $N_1(t)$, which is calculated using the normal rate of growth a_1.With a normal rate of growth of a_2, $N_2(t)$ represents the population density of the EnemyHost at the given time. The Population density of the Commensal Species is denoted by $N_3(t)$, Which Corresponds to the normal rate of growth a_3. Here a_{ii} represents the rate of decline of N_i resulting from insufficient N_i resources where i =1,2,3.a_{12} can be interpreted as the rate of decrease of the Ammesnal (N_1) that is caused by inhibition by the EnemyHost (N_2). The rate of commensal's expansion due to its effective attacks on the Ammesnal species (N_1) is shown by symbol a_{31}.$K_i(=a_i/a_{ii})$ is a representation of the carrying capabilities of N_i, where i = 1, 2, 3, respectively. The expression $P(=a_{12}/a_{11})$ is the representation of the Prey-Ammesnal suffering rate's co-efficient. The commensal coefficient, denoted by Q $(=a_{31}/a_{33})$, is presented here.

It is presumed that none of these variables or parameters under consideration have a negative value.

2 .Basic Equations:

Ecological (AEC) model equations for numerous interacting three species are represented by non-linear ordinary differential equations with variable rates that are proportional to population sizes.

(i) The growth rate equation for Ammensal species (N_1):

$$\frac{dN_1}{dt} = a_{11}N_1\left((1-m_1)K_1 - N_1 - PN_2\right) \tag{2.1}$$

(ii) The growth rate equation for EnemyHost species (N_2):

$$\frac{dN_2}{dt} = a_{22}N_2\left((1-m_2)K_2 - N_2\right) \tag{2.2}$$

(iii) The growth rate equation for Commensal species (N_3):

$$\frac{dN_3}{dt} = a_{33}N_3\left((1-m_3)K_3 - N_3 + QN_2\right) \tag{2.3}$$

3. Global Stability of Ecological (AEC) Model:

System stability is examined with regard to global stability using the Routh-Hurwitz criteria, Lyapunov's theorem, and diffusive analysis.

Theorem(3.1): The globally asymptotically stability of this system occurs at the positive equilibrium $E^*\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ of system (2.1)-(2.3).

Proof: With the help of suitable Lyapunov function V(t), The global stability nature at the positive equilibrium point $E^*\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ can be identified. Here

$$V(t) = \left(N_1 - \overline{N_1} - \overline{N_1}\ln\left(\frac{N_1}{\overline{\overline{N_1}}}\right)\right) + l_1\left(N_2 - \overline{N_2} - \overline{N_2}\ln\left(\frac{N_2}{\overline{\overline{N_2}}}\right)\right) + l_2\left(N_3 - \overline{N_3} - \overline{N_3}\ln\left(\frac{N_3}{\overline{\overline{N_3}}}\right)\right),$$

where $l_1, l_2 > 0$.

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N_1}}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{N_2 - \overline{N_2}}{N_2}\right)\frac{dN_2}{dt} + l_2\left(\frac{N_3 - \overline{N_3}}{N_3}\right)\frac{dN_3}{dt}$$

$$\frac{dV}{dt} < -\left[(a_{11} + \frac{a_{12}}{2})(N_1 - \overline{N_1})^2 + (1 - \frac{a_{32}}{2a_{33}} + \frac{a_{12}}{2})(N_2 - \overline{N_2})^2 + (1 - \frac{a_{32}}{2a_{33}})(N_3 - \overline{N_3})^2\right]$$

cleary it is noticed that the relation $a_{32} < a_{33}$ *holds* $(\therefore\ 1 - \frac{a_{32}}{2a_{33}} > 0)$

$$\Rightarrow \frac{dV}{dt} < 0$$

Clearly, $V'(t) < 0$, hence the non-diffusive system is globally asymptotically stable by Lyapunov Theorem.

Theorem (3.2): The diffusive equations of Ammensal- EnemyHost - Commensal three species ecosystem are stable if and only if $\beta_1 > 0,\ \beta_2 > 0,\ \ \beta_3 > 0,\ \beta_1\beta_2 > \beta_3$

$$where\ \beta_1 = s_1 + s_2 + s_3 > 0;\quad \beta_2 = s_1s_2 + s_2s_3 + s_3s_1 > 0;\quad \beta_3 = s_1s_2s_3 > 0$$

$$here\ s_1 = A_{11}\overline{N}_1 + \mu^2 D_1 > 0, s_2 = A_{22}\overline{N}_2 + \mu^2 D_2 > 0, s_3 = A_{33}\overline{N}_3 + \mu^2 D_3 > 0$$

Proof:

The diffusive equations of Ammensal- EnemyHost - Commensal three species system are considered as

$$\frac{\partial N_1}{\partial t} = (1-m_1)a_1N_1 - a_{11}N_1^{\ 2} - a_{12}N_1N_2 + D_1\frac{\partial^2 N_1}{\partial v^2} \tag{3.1}$$

$$\frac{\partial N_2}{\partial t} = (1-m_2)a_2N_2 - a_{22}N_2^{\ 2} + D_2\frac{\partial^2 N_2}{\partial v^2} \tag{3.2}$$

$$\frac{\partial N_3}{\partial t} = (1-m_3)a_3N_3 - a_{33}N_3^{\ 2} + a_{32}N_3N_2 + D_3\frac{\partial^2 N_3}{\partial v^2} \tag{3.3}$$

Under this spatio-temporal dynamical system,

$N_1(\mathrm{u},t),\ N_2(\mathrm{u},t)$ and $N_3(\mathrm{u},t)$ in $0 \le u \le L, L > 0$ are considered as

$$\frac{\partial N_1(0,t)}{\partial t} = \frac{\partial N_1(L,t)}{\partial t} = \frac{\partial N_2(0,t)}{\partial t} = \frac{\partial N_2(L,t)}{\partial t} = \frac{\partial N_3(0,t)}{\partial t} = \frac{\partial N_3(L,t)}{\partial t} = 0$$

After linearization of the system (3.1)-(3.3) with

$$N_1 = \overline{N_1} + u_1\ ; N_2 = \overline{N_2} + u_2\ ; N_3 = \overline{N_3} + u_3$$

$$\frac{\partial N_1}{\partial t} = (1-m_1)a_1\left(\overline{N_1} + u_1\right) - a_{11}\left(\overline{N_1} + u_1\right)^2 - a_{12}\left(\overline{N_1} + u_1\right)\left(\overline{N_2} + u_2\right) + D_1\frac{\partial^2 N_1}{\partial v^2}$$

$$= (1-m_1)a_1\overline{N_1} + (1-m_1)a_1u_1 - a_{11}\overline{N_1}^2 - a_{11}u_1^{\ 2} - 2a_{11}\overline{N_1}u_1$$

$$-a_{12}\overline{N_1}\,\overline{N_2} - a_{12}\overline{N_1}u_2 - a_{12}\overline{N_2}u_1 - a_{12}u_1u_{12} + D_1\frac{\partial^2 N_1}{\partial v^2}$$

From (3.1) – (3.3)

$$\frac{\partial u_1}{\partial t} = -A_{11}\overline{N_1}u_1 - a_{12}\overline{N_1}u_2 + D_1\frac{\partial^2 N_1}{\partial v^2}\ Where\ A_{11} = 2a_{11}$$

$$\frac{\partial u_2}{\partial t} = -A_{22}\overline{N_2}u_2 + D_2\frac{\partial^2 N_2}{\partial v^2}\ Where\ A_{22} = 2a_{22}$$

ijates
ISSN 2348 - 7550

$$\frac{\partial u_3}{\partial t} = -A_{33}\,\overline{N_3}u_3 + a_{32}\,\overline{N_3}u_2 + D_3\frac{\partial^2 N_3}{\partial v^2} \text{ where } A_{33} = 2a_{33}$$

Solutions of above system is

$u_1(v,t) = \alpha_1 e^{\lambda t}\cos\mu v\,;\, u_2(v,t) = \alpha_2 e^{\lambda t}\cos\mu v\,;\, u_3(v,t) = \alpha_3 e^{\lambda t}\cos\mu v$.

$$\frac{\partial u_1}{\partial t} = -A_{11}\,\overline{N_1}u_1 - a_{12}\,\overline{N_1}u_2 - \mu^2 u_1 D_1 = f_1\left(u_1, u_2, u_3\right) \tag{3.4}$$

$$\frac{\partial u_2}{\partial t} = -A_{22}\,\overline{N_2}u_2 - \mu^2 u_2 D_2 = f_2\left(u_1, u_2, u_3\right) \tag{3.5}$$

$$\frac{\partial u_3}{\partial t} = -A_{33}\,\overline{N_3}u_3 + a_{32}\,\overline{N_3}u_2 - \mu^2 u_3 D_3 = f_3\left(u_1, u_2, u_3\right) \tag{3.6}$$

$Characteristic equation is \left|A - \lambda I\right| = 0$

$$i.e \begin{vmatrix} -A_{11}\overline{N}_1 - \mu^2 D_1 - \lambda & -a_{12}\overline{N}_1 & 0 \\ 0 & -A_{22}\overline{N}_2 - \mu^2 D_2 - \lambda & 0 \\ 0 & a_{32}\overline{N_3} & -A_{33}\overline{N}_3 - \mu^2 D_3 - \lambda \end{vmatrix} = 0$$

$Let\; -A_{11}\overline{N}_1 - \mu^2 D_1 = -s_1 \; where\; s_1 = A_{11}\overline{N}_1 + \mu^2 D_1 > 0$

$Let\; -A_{22}\overline{N}_2 - \mu^2 D_2 = -s_2 \;\; where\; s_2 = A_{22}\overline{N}_2 + \mu^2 D_2 > 0$

$Let\; -A_{33}\overline{N}_3 - \mu^2 D_3 = -s_3 \; where\; s_3 = A_{33}\overline{N}_3 + \mu^2 D_3 > 0$

$$\Rightarrow \begin{vmatrix} -s_1 - \lambda & -a_{12}\overline{N}_1 & 0 \\ 0 & -s_2 - \lambda & 0 \\ 0 & a_{32}\overline{N_3} & -s_3 - \lambda \end{vmatrix} = 0 \tag{3.7}$$

$$\Rightarrow \lambda^3 + \left(s_1 + s_2 + s_3\right)\lambda^2 + \left(s_1 s_2 + s_2 s_3 + s_3 s_1\right)\lambda + s_1 s_2 s_3 = 0$$

From the system (3.4)-(3.6).

$Let\;\; \beta_1 = s_1 + s_2 + s_3 > 0; \;\; \beta_2 = s_1 s_2 + s_2 s_3 + s_3 s_1; \;\; \beta_3 = s_1 s_2 s_3 > 0$

$consider\; \beta_1\beta_2 - \beta_3 = \left[\left(s_1 + s_2 + s_3\right)\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right) - s_1 s_2 s_3\right]$

$= s_1^2 s_2 + s_1^2 s_3 + s_1 s_2^2 + s_2^2 s_3 + s_2 s_3^2 + s_1 s_3^2 + 2 s_1 s_2 s_3 > 0$

$clearly\; 1 > 0,\;\; \beta_1 > 0,\; \beta_2 > 0,\;\;\; \beta_3 > 0,\;\; \beta_1\beta_2 > \beta_3$

According to the Routh-Hurwitz criterion, it is only the case that all of the Eigen values of (3.7) have negative parts when $\beta_1 > 0,\; \beta_2 > 0,\;\;\; \beta_3 > 0,\;\; \beta_1\beta_2 > \beta_3$

Hence, it is stable.

Theorem (3.3):

In the case of non-diffusive effect, the diffusive model of the considered three species Ecological (AEC) system is globally asymptotically stable at the internal equilibrium point $E^*\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ under zero flux boundary conditions.

Proof:-

Define the function $\gamma_1(t)=\int_0^R \gamma(N_1,N_2,N_3)dv$

Let us define Lyapunov function

$$\gamma_1(N_1,N_2,N_3)=\left[(N_1-\overline{N_1})-\overline{N_1}\ln(\frac{N_1}{\overline{N_1}})\right]+l_1\left[(N_2-\overline{N_2})-\overline{N_2}\ln(\frac{N_2}{\overline{N_2}})\right]+l_2\left[(N_3-\overline{N_3})-\overline{N_3}\ln(\frac{N_3}{\overline{N_3}})\right]$$

Differentiating γ_1 along with the results of the diffusive model with respect to time (3.1)-(3.3) we get

$$\gamma_1^1(t)=\int_0^R\left(\frac{\partial v}{\partial N_1}.\frac{\partial N_1}{\partial t}+\frac{\partial v}{\partial N_2}.\frac{\partial N_2}{\partial t}+\frac{\partial v}{\partial N_3}.\frac{\partial N_3}{\partial t}\right)dv=I_1+I_2$$

Where $I_1=\int_0^R\frac{d\gamma}{dt}dv$ and $I_2=\int_0^R\left(D_1\frac{\partial\gamma}{\partial N_1}\frac{\partial^2 N_1}{\partial v^2}+D_2\frac{\partial\gamma}{\partial N_2}\frac{\partial^2 N_2}{\partial v^2}+D_3\frac{\partial\gamma}{\partial N_3}\frac{\partial^2 N_3}{\partial v^2}\right)dv$

Using the establish the result of B.Dubey & J.Hussain [2],

$$I_2=-D_1\int_0^R\frac{\overline{N_1}}{N_1^2}\left(\frac{\partial N_1}{\partial v}\right)^2dv-D_2\int_0^R\frac{\overline{N_2}}{N_2^2}\left(\frac{\partial N_2}{\partial v}\right)^2dv-D_3\int_0^R\frac{\overline{N_3}}{N_3^2}\left(\frac{\partial N_3}{\partial v}\right)^2dv$$

It is clearly identified that if $I_1<0$ then $\frac{d\gamma_1}{dt}$ *is negative* (<0)

As a result, the system can be said to be asymptotically stable globally

4.Conclusions :

This study concludes the following after doing an in-depth review of an ecosystem consisting of three species Ecological (AEC) Model (an Ammensal, an EnemyHost, and a Commensal species):

(i).The system is globally stable as evidenced by constructing a sufficient Lyapunov function.

(ii). Diffusion analysis provides a fruitful stage for addressing the system's stability.

5.Acknowledgments:

The present authors are grateful to '**Dr. Nazeer Shaik garu**, Principal, Bapatla Engineering College, Bapatla, **Our Magnanimous Management, Bapatla Education Society** and **HOD & Faculty Members**, Dept. of Mathematics, Bapatla Engineering College' for their constant encouragement and valuable support for successfully completing PG (M.SC-Maths) project **'A Mathematical Study on Some Special Aspects of A Peculiar Ecological Model'**

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