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Fixed Point Theorems in Fuzzy Metric Spaces

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Abstract:

Main purpose of this research paper is to prove fixed point theorems in Fuzzy Metric spaces ,which satisfy some rational inequality.we also explain some related results which satisfy some integral type inequality,that is the application of the main result.

Keywords: *Fuzzy Metric Space, Rational inequality, Integral Type inequality.*

1. Introduction

In 1965,the concept of fuzzy set was introduced by Zadeh [3].After that many other mathematician developed the theory of fuzzy sets and applications. Recently,many researchers have also studied the fixed point theory in the fuzzy metrics and have studied for fuzzy mappings which opened anavenue for further researches. In 1975, Kramosil and Michalek [5] defined the concept of fuzzy metric spaces. In 1988, Mariusz Grabiec [4] extended fixed point theorem of Banach . A number of fixed point theorem have been obtained by various authors in fuzzy metric space by using the concept of compatible map, implicit relation, weakly compatible map, R weakly compatible map. Also R.K. Saini and Vishal Gupta [9, 10] gave some fixed points theorems. The present paper prove fixed point theorem in fuzzy metric space that satisfy a contraction condition.

2. Preliminaries

In this section, we define some important definition and results which are used in this paper.

Definition 2.1 ([3]): Let Z be any set. A fuzzy set F in Z is a function with domain W and values in [0, 1].

Definition 2.2 ([2]): A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norms if ([0, 1]), ∗) is an abelian topological monoid with the unit 1 such that $p * q \leq r * s$ whenever $p \leq r$ and $q \leq s$ for all p ,q, r ,s ∈ [0, 1].

Definition2.3 ([5]): A triplet (Z, F, ∗) is a fuzzy metric space if Z is a an arbitrary set, ∗ is continuous t-norm and F is a fuzzy set on $F^2 \times (0, \infty)$ satisfying the following conditions, for all x, y, z ∈ Z, such that t, s ∈ (0, ∞).

1. $F(x, y, t) > 0$
2. $F(x, y, t) = 1$ iff $x = y$
3. $F(x, y, t) = F(y, x, t)$
4. $F(x, y, t) * F(y, z, s) \leq F(x, z, t + s)$.
5. $F(x, y, .) : [0, \infty) \to [0, 1]$ is continuous.

Thus F is called a fuzzy metric on Z and F(x, y, t) denotes the degree of nearness between x and y with respect to t.

Definition2.4 ([4]): Let (Z, F, ∗) is a fuzzy metric space then a sequence $[x_n] \in Z$ is said to be convergent to a point $x \in Z$ if $\lim_{n\to\infty} F(x_n, x, t) = 1$ for all $t > 0$.

Definition2.5 ([4]): Let (Z, F, ∗) is a fuzzy metric space then a sequence $[x_n] \in Z$ is called Cauchy sequence if $\lim_{n\to\infty} Z(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

Definition 2.6 ([4]): Let (Z, F, ∗) is a fuzzy metric space then an Fuzzy Metric space in which every Cauchy sequence is convergent is called complete. It is called compact, if every sequence contains a convergent subsequence.

Lemma.1 ([4]): For all, x, y ∈ Z, F(x, y, ·) is non-decreasing.

Lemma.2 ([11]): If there exist $k \in (0, 1)$ such that $F(x, y, kt) \geq F(x, y, t)$ for all x, y ∈ Z and $t \in (0, \infty)$, then $x = y$.

Now we prove our main result.

3. Main Results

Let us define $\Phi = \{\phi/\phi : [0, 1] \to [0, 1]\}$ is a continuous function such that $\phi(1) = 1$, $\phi(0) = 0$, $\phi(\alpha) > \alpha$ for each $0 < \alpha < 1$.

Theorem 3.1: Let (Z, F, ∗) be a complete fuzzy metric space and $f : Z \to Z$ be a mapping satisfying

$$F(x, y, t) = 1 \tag{1.1}$$

and

$$F(fx, fy, kt) \geq \phi\{\lambda(x, y, t)\}$$

Where

$$\lambda(x, y, t) = min\left\{\frac{F(y, fy, t)[1 + F(x, fx, t)]}{[1 + F(x, y, t)]},\ F(x, y, t)\right\} \tag{1.3}$$

for all $x, y \in Z$, $k \in (0, 1)$, $\phi \in \Phi$. Then f has a unique fixed point.

Proof: Since $\phi \in \Phi$. This implies that $\phi(\alpha) > \alpha$ for each $0 < \alpha < 1$. Thus from above condition

$$F(fx, fy, kt) \geq \phi\{\lambda(x, y, t)\} \geq \lambda(x, y, t) \tag{1.2}$$

Let us consider $x \in Z$ be any arbitrary point in Z. Now construct a sequence $[x_n] \in Z$ such that $fx_n = x_{n+1}$ for all $n \in N$.

Claim: $\{x_n\}$ is a Cauchy sequence.

Let us take $x = x_{n-1}$ and $y = x_n$ in (1.2), we get

$$F(x_n, x_{n+1}, kt) = (fx_{n-1}, fx_n, kt) \geq \lambda(x_{n-1}, x_n, t) \tag{1.4}$$

Now

$$\lambda(x_{n-1}, x_n, t) = min\left\{\frac{F(x_n, fx_n, t)[1 + F(x_{n-1}, fx_{n-1}, t)]}{[1 + F(x_{n-1}, x_n, t)]},\ F(x_{n-1}, x_n, t)\right\}$$

$$\lambda(x_{n-1}, x_n, t) = min\left\{\frac{F(x_n, fx_{n+1}, t)[1 + F(x_{n-1}, fx_n, t)]}{[1 + F(x_{n-1}, x_n, t)]},\ F(x_{n-1}, x_n, t)\right\}$$

$$\Rightarrow \lambda(x_{n-1}, x_n, t)$$

$$= min\{F(x_n, x_{n+1}, t), F(x_{n-1}, x_n, t)\}$$

Now if $F(x_n, x_{n+1}, t) \leq F(x_{n-1}, x_n, t)$, then from equation (1.4)

$$F(x_n, x_{n+1}, kt) \geq F(x_n, x_{n+1}, t)$$

Hence from lemma (2), our claim follows immediately. Now suppose $F(x_n, x_{n+1}, t) \geq F(x_{n-1}, x_n, t)$ then again from equation (1.4),

$$F(x_n, x_{n+1}, kt) \geq F(x_{n-1}, x_n, t)$$

Now by simple induction, for all n and t >o, we get

$$F(x_n, x_{n+1}, kt) \geq F(x, x_1, \frac{t}{k^{n-1}}) \quad (1.5)$$

Now for any positive integer 's', we have

$$F(x_n, x_{n+s,} t) \geq F(x_n, x_{n+1}, \frac{t}{s})*.....(s)....* F\times (x_{n+p-1}, x_{n+p}, \frac{t}{s})$$

By using equation (1.5), we get

$$F(x_n, x_{n+s,} t) \geq F(x, x_1, \frac{t}{sk^n})*.....(s)....* F\times (x, x_1, \frac{t}{sk^n}$$

Now taking $\lim_{n\to\infty}$ and using (1.1), we get

$$\lim_{n\to\infty} F(x_n, x_{n+s,} t) = 1 \quad (1.6)$$

This implies, $\{x_n\}$ is a Cauchy sequence. Call the limit υ.

Claim:υ is a fixed point of f.

Consider

$$F(\upsilon, f\upsilon, t) \geq F(fx_n, f\upsilon, t) * F(\upsilon, x_{n+1}, t)$$

$$\geq \lambda\left(x_n, \upsilon, \frac{t}{2k}\right) * M'(\upsilon, x_{n+1}, t) \quad (1.7)$$

Now

$$\boldsymbol{\lambda\left(x_n, v, \frac{t}{2k}\right) = min\left\{\frac{F\left(v, fv, \frac{t}{2k}\right)[1 + F(x_n, fx_n, \frac{t}{2k})]}{[1 + F(v, x_n, \frac{t}{2k})]},\quad F(v, x_n, \frac{t}{2k})\right\}}$$

Taking $\lim_{n\to\infty}$ in above inequality and using (1.1), we get

$$\boldsymbol{\lambda\left(v, v, \frac{t}{2k}\right) = min\left\{F\left(v, fv, \frac{t}{2k}\right), 1\right\}}$$

Now if $\boldsymbol{F\left(v, fv, \frac{t}{2k}\right) \geq 1}$ then $\lambda\left(\upsilon, \upsilon, \frac{t}{2k}\right) = 1$

Therefore from (1.7) and using definition 3, we get υ is a fixed point of f.

Now if $\boldsymbol{F\left(v, fv, \frac{t}{2k}\right) \geq 1}$ then $\boldsymbol{\lambda\left(v, v, \frac{t}{2k}\right) = F\left(v, fv, \frac{t}{2k}\right)}$.

Hence from equation (1.7), we get

$$\boldsymbol{F(v, fv, t) \geq F\left(v, fv, \frac{t}{2k}\right) * F(x_{n+1}, v, t)} \quad (1.8)$$

Now taking $\lim_{n\to\infty}$in (1.8) and using equation (1.1) and lemma (2), we get fυ = υ.

Uniqueness: Now we show that υ is a unique fixed point of f. Suppose not, then there exist a point ω∈ Z such that fω = ω. Consider

$$1 \leq F(w, v, t) = F(fw, v, t) \geq \lambda\left(w, v, \frac{t}{k}\right) \quad (1.9)$$

where

$$\boldsymbol{\lambda\left(w, v, \frac{t}{k}\right) = min\left\{\frac{F\left(v, fv, \frac{t}{k}\right)[1 + F(w, fw, \frac{t}{k})]}{[1 + F(w, v, \frac{t}{k})]}, \quad F(w, v, \frac{t}{k})\right\}}$$

$$\boldsymbol{\lambda\left(w, v, \frac{t}{k}\right) = min\left\{\frac{F\left(v, fv, \frac{t}{k}\right)[1 + F(w, w, \frac{t}{k})]}{[1 + F(w, v, \frac{t}{k})]}, \quad F(w, v, \frac{t}{k})\right\}}$$

$$\boldsymbol{\lambda\left(w, v, \frac{t}{k}\right) = min\{\frac{2}{\left[1 + F\left(w, v, \frac{t}{k}\right)\right]}, F(w, v, \frac{t}{k})}$$

$$= \min\{1, 1\}$$

$$\Rightarrow \lambda\left(\omega, \upsilon, \frac{t}{k}\right) = 1 \quad (1.10)$$

Use it in (1.9), we get ω = υ. Thus υ is unique fixed point of f. This completes the proof of Theorem 1.

Theorem 3.2: Let (X, F, ∗) be a complete fuzzy metric space and f: Z→Z be a mapping satisfying

$$F(x, y, t) = 1 \quad (2.1)$$

and

$$F(fx, fy, kt) \geq \lambda(x, y, t) \quad (2.2)$$

Where

$$\lambda(x,y,t) = min\left\{\frac{F(y,fy,t)[1+F(x,fx,t)]}{[1+F(x,y,t)]}, F(x,y,t)\right\} \quad \textbf{(2.3)}$$

for all x, y ∈ Z and k ∈ (0, 1). Then f has a unique fixed point.

Proof: Let us consider x ∈ Z be any arbitrary point in Z. Now construct a sequence $[x_n]$ ∈ Z such that $fx_n = x_{n+1}$ for all n ∈N.

Claim: $\{x_n\}$ is a Cauchy sequence.

Let us take $x = x_{n-1}$ and $y = x_n$ in (2.2), we get

$$F(x_n, x_{n+1}, kt) = (fx_{n-1}, fx_n, kt) \geq \lambda(x_{n-1}, x_n, t) \quad (2.4)$$

Now

$$\lambda(x_{n-1}, x_n, t) = min\left\{\frac{F(x_n, fx_n, t)[1+F(x_{n-1}, fx_{n-1}, t)]}{[1+F(x_{n-1}, x_n, t)]}, \quad F(x_{n-1}, x_n, t)\right\}$$

$$\lambda(x_{n-1}, x_n, t) = min\left\{\frac{F(x_n, fx_{n+1}, t)[1+F(x_{n-1}, fx_n, t)]}{[1+F(x_{n-1}, x_n, t)]}, \quad F(x_{n-1}, x_n, t)\right\}$$

$\Rightarrow\lambda(x_{n-1}, x_n, t)$

$= \min\{F(x_n, x_{n+1}, t), F(x_{n-1}, x_n, t)\}$

Now if $F(x_n, x_{n+1}, t) \leq F(x_{n-1}, x_n, t)$, then from equation (2.4)

$$F(x_n, x_{n+1}, kt) \geq F(x_n, x_{n+1}, t)$$

Hence from lemma (2), our claim follows immediately. Now suppose $F(x_n, x_{n+1}, t) \geq F(x_{n-1}, x_n, t)$ then again from equation (2.4),

$$F(x_n, x_{n+1}, kt) \geq F(x_{n-1}, x_n, t)$$

Now by simple induction, for all n and t >o, we get

$$F(x_n, x_{n+1}, kt) \geq F(x, x_1, \frac{t}{k^{n-1}}) \quad (2.5)$$

Now for any positive integer 's', we have

$$F(x_n, x_{n+s,}t) \geq F(x_n, x_{n+1}, \frac{t}{s})*\ldots..(s)\ldots.* F\times (x_{n+p-1}, x_{n+p}, \frac{t}{s})$$

By using equation (2.5), we get

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$F(x_n, x_{n+s,} t) \geq F(x, x_1, \frac{t}{sk^n}$)*…..(s)….* F× $(x, x_1, \frac{t}{sk^n}$

Now taking $\lim_{n\to\infty}$ and using (2.1), we get

$$\lim_{n\to\infty} F(x_n, x_{n+s,} t) = 1 \tag{2.6}$$

This implies, $\{x_n\}$ is a Cauchy sequence. Call the limit υ.

Claim:υ is a fixed point of f.

Consider

$$F(\upsilon, f\upsilon, t) \geq F(fx_n, f\upsilon, t) * F(\upsilon, x_{n+1}, t)$$

$$\geq \lambda\left(x_n, \upsilon, \frac{t}{2k}\right) * M'(\upsilon, x_{n+1}, t) \tag{2.7}$$

Now

$$\boldsymbol{\lambda\left(x_n, v, \frac{t}{2k}\right) = min\left\{\frac{F\left(v, fv, \frac{t}{2k}\right)[1 + F(x_n, fx_n, \frac{t}{2k})]}{[1 + F(v, x_n, \frac{t}{2k})]},\ \ F(v, x_n, \frac{t}{2k})\right\}}$$

Taking $\lim_{n\to\infty}$ in above inequality and using (1.1), we get

$$\boldsymbol{\lambda\left(v, v, \frac{t}{2k}\right) = \ min\left\{\ F\left(v, fv, \frac{t}{2k}\right), 1\right\}}$$

Now if $\boldsymbol{F\left(v, fv, \frac{t}{2k}\right) \geq 1}$ then $\lambda\left(\upsilon, \upsilon, \frac{t}{2k}\right) = 1$

Therefore from (2.7) and using definition 3, we get υ is a fixed point of f.

Now if $\boldsymbol{F\left(v, fv, \frac{t}{2k}\right) \geq 1}$ then $\boldsymbol{\lambda\left(v, v, \frac{t}{2k}\right) = F\left(v, fv, \frac{t}{2k}\right)}$.

Hence from equation (1.7), we get

$$\boldsymbol{F(v, fv, t) \geq F\left(v, fv, \frac{t}{2k}\right) * F(x_{n+1}, v, t)} \tag{2.8}$$

Now taking $\lim_{n\to\infty}$in (2.8) and using equation (2.1) and lemma (2), we get fυ = υ.

Uniqueness: Now we show that υ is a unique fixed point of f. Suppose not, then there exist a point ω∈ Z such that fω = ω. Consider

$$1 \leq F(w, v, t) = F(fw, v, t) \geq \lambda\left(w, v, \frac{t}{k}\right) \tag{2.9}$$

where

$$\lambda\left(w,v,\frac{t}{k}\right) = min\left\{\frac{F\left(v,fv,\frac{t}{k}\right)[1+F(w,fw,\frac{t}{k})]}{[1+F(w,v,\frac{t}{k})]}, \quad F(w,v,\frac{t}{k})\right\}$$

$$\lambda\left(w,v,\frac{t}{k}\right) = min\left\{\frac{F\left(v,fv,\frac{t}{k}\right)[1+F(w,w,\frac{t}{k})]}{[1+F(w,v,\frac{t}{k})]}, \quad F(w,v,\frac{t}{k})\right\}$$

$$\lambda\left(w,v,\frac{t}{k}\right) = min\{\frac{2}{\left[1+F\left(w,v,\frac{t}{k}\right)\right]}, F(w,v,\frac{t}{k})$$

$$= \min\{1, 1\}$$

$$\Rightarrow \lambda\left(\omega,\upsilon,\frac{t}{k}\right) = 1 \qquad (2.10)$$

Use it in (2.9), we get ω = υ. Thus υ is unique fixed point of f. This completes the proof of Theorem .

4. Applications

In this section, we gives some application related to our results. Let us define Ψ: [0,∞] → [0,∞], as $\Psi(t) = \int_0^t \varphi(t)dt \ \forall t > 0$, be a non-decreasing and continuous function. Moreover, for each ε> 0, φ(ε) > 0. Also implies that φ(t) = 0 iff t = 0.

Theorem 4.1: Let (Z, F, ∗) be a complete fuzzy metric space and f : Z→ Z be a mapping satisfying

F(x, y, t) = 1

$$\int_0^{F(fx,fy,kt)} \varphi(t)dt \geq \int_0^{\lambda(x,y,t)} \varphi(t)dt$$

where

$$\lambda(x,y,t) = min\left\{\frac{F(y,fy,t)[1+F(x,fx,t)]}{[1+F(x,y,t)]}, F(x,y,t)\right\}$$

for all x, y ∈ Z, φ∈Ψ and k ∈ (0, 1). Then f has a unique fixed point.

Proof: By taking φ(t) = 1 and applying Theorem 3.2, we obtain the result.

Theorem 4.2: Let (Z, F, ∗) be a complete fuzzy metric space and f : Z→Z be a mapping satisfying

F(x, y, t) = 1

$$\int_0^{F(fx,fy,kt)} \varphi(t)dt \geq \phi\{\int_0^{\lambda(x,y,t)} \varphi(t)dt\}$$

where

$$\boldsymbol{\lambda(x,y,t) = min\left\{\frac{F(y,fy,t)[1+F(x,fx,t)]}{[1+F(x,y,t)]}, F(x,y,t)\right\}}$$

for all x, y ∈ Z, φ∈Ψ, k ∈ (0, 1) and ϕ∈Φ. Then f has a unique fixed point.

Proof: Since ϕ(α) >α for each 0 <α< 1, therefore result follows immediately from Theorem 4.3.

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