STRONG AND WEAK DOMINATING-χ-COLOR NUMBER OF k -PARTITE GRAPH

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ABSTRACT

Let G = (V, E) be a graph. A graph G is k -colorable if it has a proper k - coloring. The chromatic number $\chi(G)$ is the least k such that G is k - colorable. In this paper, we define strong and weak dominating χ color number of a graph G as the maximum number of color classes which are strong and weak dominating sets of G, and are denoted by s $d_\chi(G)$ and $wd_\chi(G)$ respectively, where the maximum is takenover all χ -coloring of G. Also we discuss the strong and weak dominating-χ-color number of a k-Partite graph.

Keywords -- *Dominating-χ-Color Number, K- Partite Graph, Middle Graph, Strong And Weak Dominating-χ -Color Number*

I. INTRODUCTION

Let G = (V, E) be a simple, connected, finite, undirected graph. The order and size of G are denoted by n and m respectively [1]. In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If $\chi(G) = k$, we say that G is k-chromatic[1].

A set $D \subseteq V$ is a dominating set of G, if for every vertex $x \in V-D$ there is a vertex $y \in D$ with $xy \in E$ and D is said to be strong dominating set of G, if it satisfies the additional condition $\deg(x) \leq \deg(y)$ [2]. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A set $S \subseteq V$ is called weak dominating set of G if for every vertex $u \in V-S$, there exists vertex $v \in S$ such that $uv \in E$ and $\deg(u) \geq \deg(v)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set and was introduced by Sampathkumar and Pushpa Latha (Discrete Math. 161(1996)235-242)[3].

II. TERMINOLOGIES

We start with notation and more formal definitions.

Let G=(V(G),E(G)) bea graph with n=|V(G)| and m=|E(G)|. For any vertex $v \in V(G)$, the openneighborhood of v is the set$N(v) = \{u| \ uv \in E\ (G)\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set $S \subseteq V(G)$, $N(S)=\cup_{v\in s} N(v) - S$ and $N[S] = N(S) \cup S$.A set S is a dominating set if N[S]=V(G).The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$[4].

Definition2.1

The Middle graph of G, denoted by M(G) is defined as follows.

The vertex set of M(G) is V (G)∪E(G). Two vertices x, y in thevertex set of M(G) are adjacent in M(G) in case one of the following holds.

1. x, y are in E(G) and x, y are adjacent in G.
2. x is in V(G), y is in E(G) and x,y are incident in G [3].

Definition2.2

Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2, \ldots, V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C whichare dominating sets of G. Then $d_\chi(G) = \max_C d_C$ where the maximum is takenover all the k-colorings of G, is called the dominating-χ-color number of G.[5]

Definition2.3

Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2, \ldots, V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C whichare dominating sets of G. Then $md_\chi(G) = \min_C d_C$ where the minimum is taken over all the k-colorings of G, is called the minimum dominating-χ-color number of G.[6]

III. MAIN RESULTS

Definition3.1

Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2 \ldots V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C whicharestrong dominating sets of G. Then $sd_\chi(G) = \max_C d_C$ where the maximum is takenover all the k-colorings of G, is called the Strong dominating-χ-color number of G.

Definition 3.2

Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2 \ldots V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C whichareweak dominating sets of G. Then $wd_\chi(G) = \max_C d_C$ where the maximum is takenover all the k-colorings of G, is called the Weak dominating-χ-color number of G.

Proposition 3.3Strong dominating-χ-color number of G exists for all graphs G and$0 \le sd_\chi(G) \le d_\chi(G) \le \chi(G)$.

Proposition 3.4Weak dominating-χ-color number of G exists for all graphs G and$0 \le wd_\chi(G) \le d_\chi(G) \le \chi(G)$.

Proposition 3.5Every Strong dominating-χ-color number set has atleast one vertex of maximum degree.

Proposition 3.6Every weak dominating-χ-color number set has atleast one vertex of minimum degree.

Theorem 3.7

For any graph G, $\mathbf{0} \le \frac{sd_\chi(G)+wd_\chi(G)}{2} \le d_\chi(G)$

Proof:Since $0 \le sd_\chi(G) \le d_\chi(G)$ and $0 \le wd_\chi(G) \le d_\chi(G)$, we will have $0 \le sd_\chi(G) + wd_\chi(G) \le 2d_\chi(G)$. Hence the proof

Theorem 3.8

If G is any k-partite graph with partition $V_1, V_2 \ldots V_k$,then

i) $sd_\chi(G) \le |X|$ where $X = \{ V_i / \deg(v) = \Delta(G) , v \in V_i\}$

ii) $wd_\chi(G) \le |Y|$ where $Y = \{ V_i / \deg(v) = \delta(G) , v \in V_i\}$.

Equality holds for both the conditions if G is complete k-partite.

Proof:Suppose,$sd\chi(G) > |X|$ where $X = \{ V_i / \deg(v) = \Delta(G) , v \in V_i\}$then there exists a partition V_i,which is a strong dominating-χ-color set, for some iand $V_i \notin X$. Hence by proposition 3.5, we have a vertex in V_i of maximumdegree.Which contradicts $V_i \notin X$

Similarly,we can prove for (ii).

Also, for a complete k-partite graph each partition is a dominating set.Hence there existsno partition with maximum degree which is also a strong dominating-χ- color set.Hence the theorem.

We now see the weak dominating-χ-color number for middle graphs of P_n and C_n.

Theorem 3.9

For any path P_n ,$wd_\chi(M(P_n)) = 1$, for all n.

Proof:Let $P_n : v_1, v_2, \ldots, v_n$ be a path of length n-1 and let $v_iv_{i+1} = e_i$.By the definition of middle graph, $M(p_n)$ has the vertex set $V(P_n) \times E(P_n) = \{v_i/1 \le i \le n\} \cup \{e_i/1 \le i \le n-1\}$ in which each e_i is adjacent to v_{i+1} and v_i,for i = 1,2,3,….,n-1. Also e_i is adjacent to e_{i+1}, for i = 1,2,3,….,n-1.

The color class partition is given by,

$\chi = \{\{e_1, e_3, \ldots \ldots e_{n-1}\}, \{v_1, v_2, \ldots \ldots, v_n\}, \{e_2, e_4, \ldots \ldots, e_{n-2}\}\}$, for even n.

And for odd n,$\chi = \{\{e_1, e_3, \ldots \ldots e_{n-2}\}, \{v_1, v_2, \ldots \ldots, v_n\}, \{e_2, e_4, \ldots \ldots, e_{n-1}\}\}$

Also each set in the color class partition is a dominating set. And the only colorclass partition which is a weak dominating set is $\{v_1, v_2, \ldots \ldots, v_n\}$, since eachvertex in this partition will be adjacent only to the vertex of degree greaterthan this vertex. Hence $wd_\chi(M(P_n)) = 1$.

Theorem 3.10

If G is any cycle C_n then,

$$wd_\chi(M(C_n)) = \begin{cases} 1 & if\ n\ is\ even \\ 0 & if\ n\ is\ odd \end{cases}$$

Proof: Let C_n:$v_1, v_2, \dots \dots, v_n, v_{n+1} (= v_1)$ be a path of length n and let $v_i v_{i+1} = e_i$ for i = 1,2,……, n-1 and $v_1 v_n = e_n$. By the definition of middle graph,$M(C_n)$has the vertex set $V(C_n) \times E(C_n) = \{ v_i / 1 \le i \le n\} \cup \{e_i / 1 \le i \le n\}$ in whicheach v_i is adjacent to $e_i$and e_{i-1} and each $e_i$is adjacent to v_{i+1} and v_i fori = 2,3,….. n-1 and v_1 is adjacent to e_1 and e_n, also e_n is adjacent to v_1 andv_n.

The only color class partition is given by, $\chi = \{\{e_1, e_3, \dots \dots . e_{n-1}\}, \{v_1, v_2, \dots \dots, v_n\}, \{e_2, e_4, \dots \dots, e_n\}\}$ for even n. And the only color class partition which is aweak dominating set is $\{v_1, v_2, \dots \dots, v_n\}$since each vertex in this partitionwill be adjacent only to the vertex of degree greater than this vertex. Hence$wd_\chi(M(C_n)) = 1$.

And for odd n,

$$\chi = \{\{e_1, e_3, \dots \dots . e_{n-2}, v_n\}, \{v_2, v_3, \dots \dots, v_{n-1}, e_n\}, \{e_2, e_4, \dots \dots, e_{n-1}, v_1\}\}$$

Also each set in the color class partition is a dominating set. And there is nocolor class partition which is a weak dominating set ,since there exists atleastone vertex outside the partition which is not adjacent to the vertex of minimumdegree in that partition. Hence $wd_\chi(M(C_n)) = 0$.

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