A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS IN COMPLETE METRIC SPACE USING RATIONAL INEQUALITY

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ABSTRACT

The object of this paper is to obtain common unique fixed point theorems for three and four self mappings satisfy rational inequality in complete metric space using concept of compatible mappings. Our result generalizes the results of Fisher [3, 4], Jungck [8] Singh and Chauhan [16] and Lohani and Badshah [11].

KEYWORDS: ***Complete Metric Space, Compatible Mappings, Common fixed Point***

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I. INTRODUCTION

Jungck [6], proved a common fixed point theorem for commuting mappings in 1976, which generalizes the Banach's [1] fixed point theorem in complete metric space. This result was generalized and extended in various ways by Iseki and Singh [5], Park [12], Das and Naik [2], Singh [17], Singh and Singh [18], Fisher [3], Park and Bae [13]. Recently, some common fixed point theorems of three and four commuting mappings were proved by Fisher [4], Khan and Imdad [10], Kang and Kim [9] .The result of Jungck [6] has so many applications but it has certain limitation that it requires the continuity of one of the mapping. S. Sessa [14], introduced the concept of weak commutativity and proved a common fixed point theorem for weakly commuting maps. In general, commuting mappings are weakly commuting but the converse is not necessarily true.

Further Jungck [7], introduced more generalized commutativity; known as compatibility, which is weaker than weakly commuting maps. In general, weakly commuting mappings are compatible, but converse is not true. Since various author proved a common fixed point theorem for compatible mappings satisfying contractive type conditions and continuity of one of the mapping is required.

The main purpose of this paper is to present fixed point theorems for three and four self maps satisfying a new rational condition by using the concept of compatible maps in a complete metric space. Also we generalize the results of Fisher [3, 4], Jungck [8], Singh and Chauhan [16] and Lohani and Badshah [11], by using another type of rational inequality. To illustrate our main theorems, an example is also given.

II. PRELIMINARIES:

Definition 2.1 : Two mapping S and T from a metric space (X, d) into itself, are called commuting on X, if $d(STx, TSx) = 0$ i.e. $STx = TSx$ for all x in X.

Definition 2.2 : Two mapping S and T from a metric space (X, d) into itself, are called weakly commuting on X, if $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X. Clearly, commuting mappings are weakly commuting, but converse is not necessarily true, given by following example :

Example2.1

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \rightarrow X$ by

$$Sx = \frac{x}{3-x} \quad \text{and} \quad Tx = \frac{x}{3} \text{ for all } x \text{ in } X.$$

Then for any x in X ,

$$\begin{aligned} d(STx, TSx) &= \left| \frac{x}{9-x} - \frac{x}{9-3x} \right| \\ &= \left| \frac{-2x^2}{(9-x)(9-3x)} \right| \\ &\leq \frac{x^2}{9-3x} \\ &= \left| \frac{x}{3-x} - \frac{x}{3} \right| \\ &= d(Sx, Tx) \end{aligned}$$

i.e. $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Thus S and T are weakly commuting mappings on X, but they are not commuting on X, because

$$STx = \frac{x}{9-x} < \frac{x}{9-3x} = TSx \text{ for any } x \neq 0 \text{ in } X.$$

i.e. $STx < TSx$ for any $x \neq 0$ in X.

Definition 2.3: If Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings on X, if $\lim_{m\to\infty} d(STx_m, TSx_m) = 0$, when $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X.

Clearly Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings on X, then $d(STx, TSx) = 0$ when $d(Sx, Tx) = 0$ for some x in X . Note that weakly commuting mappings are compatible, but the converse is not necessarily true.

Example2.2

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \rightarrow X$ by

$$Sx = x \quad \text{and} \quad Tx = \frac{x}{x+1} \text{ for all } x \text{ in } X.$$

Then for any x in X ,

$$STx = S(Tx) = S\left(\frac{x}{x+1}\right) = \frac{x}{x+1}$$

$$TSx = T(Sx) = T(x) = \frac{x}{x+1}$$

$$d(Sx, Tx) = \left|x - \frac{x}{x+1}\right| = \left|\frac{x^2}{x+1}\right|$$

Thus we have

$$d(STx, TSx) = \left|\frac{x}{x+1} - \frac{x}{x+1}\right|$$

$$= 0 \le \frac{x^2}{x+1} \text{ for all } x \text{ in } X.$$

$$= d(Sx, Tx)$$

i.e $d(STx, TSx) \le d(Sx, Tx)$ for all x in X.

Thus *S* and *T* are weakly commuting mappings on *X*, and then obviously *S* and *T* are compatible mappings on *X*

Example2.3

Let $X = R$ with the Euclidean metric *d*. Define *S* and $T : X \to X$ by

$Sx = x^2$ and $Tx = 2x^2$ for all x in X.

Then for any x in X ,

$$STx = S(Tx) = S(2x^2) = 4x^4$$

$TSx = T(Sx) = T(x^2) = 2x^4$ are compatible mappings on X , because

$$d(Sx, Tx) = |x^2 - 2x^2| = |-x^2| \to 0 \;\; as \;\; x \to 0$$

Then

$$d(STx, TSx) = |4x^4 - 2x^4| = 2|x^4| \to 0 \;\; as \;\; x \to 0$$

But $d(STx, TSx) \le d(Sx, Tx)$ is not true for all x in X.

Thus *S* and *T* are not weakly commuting mappings on *X*. Hence all weakly commuting mappings are compatible, but converse is not true.

Lemma 2.1 [8]. Let *S* and *T* be compatible mappings from a metric space (*X*, *d*) into itself. Suppose that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X .Then $\lim_{m\to\infty} TSx_m = Sx$, if *S* is continuous.

Now, let *P*, *S* and *T* be three mappings from a complete metric space (*X*, *d*) into itself satisfying the conditions:

$$S(X) \cup T(X) \subseteq P(X) \tag{2.1}$$

$$d(Sx, Ty) \le \left\{\alpha + \beta \frac{d(Sx, Px)}{1 + d(Px, Py)}\right\} d(Ty, Py) \tag{2.2}$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha < 1$ and $\alpha + \beta < 1$.

Then for an arbitrary point x_0 in X, by (2.1) we choose a point x_1 in X such that $Px_1 = Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2 = Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$$y_{2m+1} = Px_{2m+1} = Sx_{2m} \text{ and } y_{2m} = Px_{2m} = Tx_{2m-1} \quad (2.3)$$

Lemma 2.2 Let P, S and T be three mappings from a complete metric space (X, d) into itself satisfying the conditions (2.1) and (2.2). Then the sequence $\{y_m\}$ defined by (2.3) is a Cauchy sequence in X.

Proof: By definition (2.3) we have

$$d(y_{2m+1}, y_{2m}) = d(Sx_{2m}, Tx_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(Sx_{2m}, Px_{2m})}{1 + d(Px_{2m}, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(y_{2m+1}, y_{2m})}{1 + d(y_{2m}, y_{2m-1})}\right\} d(y_{2m}, y_{2m-1})$$

$$\leq \alpha d(y_{2m}, y_{2m-1}) + \beta d(y_{2m+1}, y_{2m})$$

i.e $d(y_{2m+1}, y_{2m}) \leq \dfrac{\alpha}{1-\beta} d(y_{2m}, y_{2m-1})$

Hence $d(y_{2m+1}, y_{2m}) \leq h\, d(y_{2m}, y_{2m-1})$

Where $h = \dfrac{\alpha}{1-\beta} < 1$

Similarly we can show that

$$d(y_{2m+1}, y_{2m}) \leq h^{2m} d(y_1, y_0)$$

For k > m , we have

$$d(y_{m+k}, y_m) \leq \sum_{i=1}^{k} d(y_{n+i}, y_{n+i-1})$$

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(y_1, y_0)$$

i.e. $d(y_{m+k}, y_m) \leq \left(\dfrac{h^n}{1-h}\right) d(y_1, y_0) \to 0$ as $n \to \infty$

Hence $\{y_m\}$ is a Cauchy's sequence in X.

III. MAIN RESULT

Theorem 3.1 Let P, S and T be three mappings from a complete metric space (X, d) into itself satisfying the conditions (2.1) and (2.2). Suppose that

(i) One of P, S and T is continuous,

(ii) The pairs (S, P) and (T, P) are compatible on X.

Then P, S and T have a unique common fixed point in X.

Proof: Let $\{y_m\}$ be the sequence in X defined by (2.3). By lemma 2.2, the sequence $\{y_m\}$ is a Cauchy sequence in X and X is complete, hence the sequence $\{y_m\}$ converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$ and $\{Tx_{2m-1}\}$ of the sequence $\{y_m\}$ also converges to u.

Now suppose that P is continuous. Since the pair (S, P) is compatible on X, lemma 2.1 gives that P^2x_{2m} and $SPx_{2m} \to Pu$ as $m \to \infty$.

By (2.2), we obtain

$$d(SPx_{2m}, Tx_{2m-1}) \le \left\{\alpha + \beta \frac{d(SPx_{2m}, PPx_{2m})}{1 + d(PPx_{2m}, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

$$\le \left\{\alpha + \beta \frac{d(SPx_{2m}, P^2x_{2m})}{1 + d(P^2x_{2m}, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Pu,u) \le \left\{\alpha + \beta \frac{d(Pu,Pu)}{1 + d(Pu,u)}\right\} d(u,u)$$

Which implies

$$d(Pu,u) \le 0$$

So that $u = Pu$.

Now from (2.2)

$$d(Su, Tx_{2m-1}) \le \left\{\alpha + \beta \frac{d(Su, Pu)}{1 + d(Pu, Px_{2m-1})}\right\} d(Tx_{2m-1}, Px_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Su,u) \le \left\{\alpha + \beta \frac{d(Su,u)}{1 + d(u,u)}\right\} d(u,u)$$

Which implies

$$d(Su,u) \le 0$$

So that $u = Su$. Since $u = Su \in S(X) \Rightarrow u = Su \in S(X) \cup T(X) \Rightarrow u = Su \in P(X)$ hence there exists a point v in X such that $u = Su = Pv$.

$$d(u,Tv) = d(Su,Tv)$$

$$\le \left\{\alpha + \beta \frac{d(Su, Pu)}{1 + d(Pu, Pv)}\right\} d(Tv, Pv)$$

$$= \left\{\alpha + \beta \frac{d(u,u)}{1 + d(u, Pv)}\right\} d(Tv, u)$$

Which implies

$(1-\alpha)d(u,Tv)\le 0$, since $\alpha<1$. So that $u=Tv$.

Hence $u=Tv=Pv$ also the pair (T, P) is compatible on X , then $d(TPv, PTv) = 0$ and so $Pu = PTv = TPv = Tu$. Thus $u = Su = Pu = Tu$.

Therefore, u is common fixed point of P, S and T. Similarly, we can also complete the proof, when S or T is continuous.

For uniqueness of u, suppose u and z, $u \neq z$, are common fixed points of P, S and T. Then by (2.2), we obtain

$$\begin{aligned} d(u,z) &= d(Su,Tz) \\ &\le \left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,Pz)}\right\}d(Tz,Pz) \\ &\le \left\{\alpha+\beta\frac{d(u,u)}{1+d(u,z)}\right\}d(z,z) \\ &\le 0 \end{aligned}$$

i.e. $d(u,z)\le 0$

which is a contradiction . Therefore $u = z$. Hence u is unique common fixed point of P, S and T. This completes the proof.

Now, let P, Q, S and T be four mappings from a complete metric space (X, d) into itself satisfying the conditions:

$$S(X)\subseteq Q(X),\ T(X)\subseteq P(X) \tag{3.1}$$

and
$$d(Sx,Ty)\le\left\{\alpha+\beta\frac{d(Sx,Px)}{1+d(Px,Qy)}\right\}d(Ty,Qy) \tag{3.2}$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$, $\alpha < 1$ and $\alpha + \beta < 1$.

Then for an arbitrary point x_0 in X, by (3.1) we choose a point x_1 in X such that $Qx_1 = Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2 = Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$$y_{2m+1} = Qx_{2m+1} = Sx_{2m} \quad \text{and} \quad y_{2m} = Px_{2m} = Tx_{2m-1} \tag{3.3}$$

Lemma 3.1 Let P, Q, S and T be four mappings from a complete metric space (X, d) into itself satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_m\}$ defined by (3.3) is a Cauchy sequence in X.

Proof: By definition (3.3) we have

$$\begin{aligned} d(y_{2m+1},y_{2m}) &= d(Sx_{2m},Tx_{2m-1}) \\ &\le \left\{\alpha+\beta\frac{d(Sx_{2m},Px_{2m})}{1+d(Px_{2m},Qx_{2m-1})}\right\}d(Tx_{2m-1},Qx_{2m-1}) \\ &\le \left\{\alpha+\beta\frac{d(y_{2m+1},y_{2m})}{1+d(y_{2m},y_{2m-1})}\right\}d(y_{2m},y_{2m-1}) \\ &\le \alpha d(y_{2m},y_{2m-1})+\beta d(y_{2m+1},y_{2m}) \end{aligned}$$

i.e $d(y_{2m+1}, y_{2m}) \le \frac{\alpha}{1-\beta} d(y_{2m}, y_{2m-1})$

Hence $d(y_{2m+1}, y_{2m}) \le h\, d(y_{2m}, y_{2m-1})$

Where $h = \frac{\alpha}{1-\beta} < 1$

Similarly we can show that

$$d(y_{2m+1}, y_{2m}) \le h^{2m} d(y_1, y_0)$$

For k > m , we have

$$d(y_{m+k}, y_m) \le \sum_{i=1}^{k} d(y_{n+i}, y_{n+i-1})$$
$$\le \sum_{i=1}^{k} h^{n+i-1} d(y_1, y_0)$$

i.e. $d(y_{m+k}, y_m) \le \left(\frac{h^n}{1-h}\right) d(y_1, y_0) \to 0$ as $n \to \infty$

Hence $\{y_m\}$ is a Cauchy's sequence in *X*.

Now we will extend theorem 3.1 for four self mappings.

Theorem 3.2 Let *P*, *Q*, *S* and *T* be mappings from a complete metric space (*X*, *d*) into itself satisfying the conditions (3.1) and (3.2). Suppose that

(i) One of *P*, *Q*, *S* and *T* is continuous, (3.4)

(ii) The pairs (*S*, *P*) and (*T*, *Q*) are compatible on *X*. (3.5)

Then P, Q, S and T have a unique common fixed point in *X*.

Proof: Let $\{y_m\}$ be the sequence in *X* defined by (3.3). By lemma (3.1), the sequence $\{y_m\}$ is a Cauchy sequence in *X* and *X* is complete, hence it converges to some point *u* in *X*. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$, $\{Tx_{2m-1}\}$ and $\{Qx_{2m-1}\}$ of the sequence $\{y_m\}$ also converges to *u*.

Now, suppose that *P* is continuous. Since the pair (*S*, *P*) is compatible on *X*, lemma (2.1) gives that P^2x_{2m} and $SPx_{2m} \to Pu$ as $m \to \infty$.

By (3.2), we obtain

$$d(SPx_{2m}, Tx_{2m-1}) \le \left\{\alpha + \beta \frac{d(SPx_{2m}, PPx_{2m})}{1 + d(PPx_{2m}, Qx_{2m-1})}\right\} d(Tx_{2m-1}, Qx_{2m-1})$$
$$\le \left\{\alpha + \beta \frac{d(SPx_{2m}, P^2x_{2m})}{1 + d(P^2x_{2m}, Qx_{2m-1})}\right\} d(Tx_{2m-1}, Qx_{2m-1})$$

Letting $m \to \infty$ and using above results we get

$$d(Pu, u) \le \left\{\alpha + \beta \frac{d(Pu, Pu)}{1 + d(Pu, u)}\right\} d(u, u)$$

Which implies

$$d(Pu, u) \le 0$$

So that $u = Pu$.

Consider

$$d(Su,Tx_{2m-1})\le\left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,Qx_{2m-1})}\right\}d(Tx_{2m-1},Qx_{2m-1})$$

Letting $m\to\infty$ and using above results we get

$$d(Su,u)\le\left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,u)}\right\}d(u,u)$$

Which implies

$$d(Su,u)\le 0$$

So that $u=Su$.

Since $S(X)\subseteq Q(X)$ and there exists a point *v* in *X*, such that $u=Su = Qv$.

Consider

$$d(u,Tv)=d(Su,Tv)$$
$$\le\left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,Qv)}\right\}d(Tv,Qv)$$
$$\le\left\{\alpha+\beta\frac{d(u,u)}{1+d(u,u)}\right\}d(Tv,u)$$

i.e. $d(u,Tv)\le\alpha d(Tv,u)$ since α < 1 .So that $u = Tv$.

Since the pair (*T*, *Q*) is compatible on *X* and $Qv = Tv = u$, then $d(TQv, QTv) = 0$ and so $Qu = QTv = TQv = Tu$.

Moreover by (3.2), we obtain

$$d(u,Qu)=d(Su,Tu)$$
$$\le\left\{\alpha+\beta\frac{d(Su,Pu)}{1+d(Pu,Qu)}\right\}d(Tu,Qu)$$
$$\le 0$$

i.e. $d(u,Qu)\le 0$. So that $u = Qu$. Thus $u = Qu = Tu = Pu = Su$.

Therefore, *u* is a common fixed point of *P*, *Q*, *S* and *T*. Similarly, we can also complete the proof, when *Q* is continuous.

Next suppose that *S* is continuous. Since the pair (*S*, *P*) is compatible on *X*, lemma (2.1) gives that S^2x_{2m} and $PSx_{2m}\to Su$ as $m\to\infty$.

By (3.2), we obtain

$$d(SSx_{2m},Tx_{2m-1})\le\left\{\alpha+\beta\frac{d(SSx_{2m},PSx_{2m})}{1+d(PSx_{2m},Qx_{2m-1})}\right\}d(Tx_{2m-1},Qx_{2m-1})$$

$$d(S^2x_{2m},Tx_{2m-1})\le\left\{\alpha+\beta\frac{d(S^2x_{2m},PSx_{2m})}{1+d(PSx_{2m},Qx_{2m-1})}\right\}d(Tx_{2m-1},Qx_{2m-1})$$

Letting $m\to\infty$ and using above results we get

$$d(Su,u)\le\left\{\alpha+\beta\frac{d(Su,Su)}{1+d(Su,u)}\right\}d(u,u)$$

Which implies

$$d(Su,u) \le 0$$

So that $u = Su$. Since $S(X) \subseteq Q(X)$ and there exists a point w in X, such that $u = Su = Qw$.

Consider

$$d(S^2x_{2m},Tw) \le \left\{\alpha + \beta \frac{d(S^2x_{2m},PSx_{2m})}{1+d(PSx_{2m},Qw)}\right\} d(Tw,Qw)$$

Letting m $\to\infty$ and using above results we get

$$d(Su,Tw) \le \left\{\alpha + \beta \frac{d(Su,Su)}{1+d(Su,Qw)}\right\} d(Tw,Qw)$$

Which implies

$$d(u,Tw) \le \alpha\, d(u,Tw)$$

So that $u = Tw$. Since the pair (T, Q) is compatible on X and $Qw = Tw = u$, then $d(QTw, TQw) = 0$ and hence $Qu = QTw = TQw = Tu$.

Moreover by (3.2) ,we have

$$d(Sx_{2m},Tu) \le \left\{\alpha + \beta \frac{d(Sx_{2m},Px_{2m})}{1+d(Px_{2m},Qu)}\right\} d(Tu,Qu)$$

Letting $m \to\infty$ and using above results we get

$$d(u,Tu) \le \left\{\alpha + \beta \frac{d(Su,u)}{1+d(u,u)}\right\} d(Tu,Tu)$$

i.e. $d(u,Tu) \le 0$.

So that $u = Tu$. Since $T(X) \subseteq P(X)$ and there exists a point z in X, such that $u = Tz = Pz$.

Moreover by (3.2), we obtain

$$\begin{aligned} d(Sz,u) &= d(Sz,Tu) \\ &\le \left\{\alpha + \beta \frac{d(Sz,Pz)}{1+d(Pz,Qu)}\right\} d(Tu,Qu) \\ &\le \left\{\alpha + \beta \frac{d(Sz,u)}{1+d(u,u)}\right\} d(u,u) \\ &\le 0 \end{aligned}$$

i.e. $d(Sz,u) \le 0$.

So that $u = Sz$. Since the pair (S, P) is compatible on X and $Sz = Pz = u$, then $d(PSz, SPz) = 0$ and hence $Pu = SPz = PSz = Su$. Therefore, u is a common fixed point of P, Q, S and T. Similarly, we can also complete the proof, when T is continuous.

For uniqueness of u, suppose u and z, $u \neq z$, are common fixed points of P, Q, S and T. Then by (3.2), we obtain

$$\begin{aligned} d(u,z) &= d(Su,Tz) \\ &\le \left\{\alpha + \beta \frac{d(Su,Pu)}{1+d(Pu,Qz)}\right\} d(Tz,Qz) \end{aligned}$$

$$\leq \left\{\alpha+\beta\frac{d(u,u)}{1+d(u,z)}\right\}d(z,z)$$

$$\leq 0$$

i.e. $d(u,z)\leq 0$

which is a contradiction .Hence $u = z$. Therefore, u is unique common fixed point of P, Q, S and T. This completes the proof.

The following corollaries follow immediately from theorem 3.2

Corollary 3.1 Let P, Q, S and T be four mappings from a complete metric space (X, d) into itself satisfying the conditions (3.1) and (3.2).Then P, Q, S and T have a unique common fixed point in X.

Corollary 3.2 Let P, Q, S and T be four mappings from a complete metric space (X, d) into itself satisfying the conditions (3.1), (3.4) , (3.5) and

$$d(Sx,Ty)\leq\left\{\alpha+\beta\frac{d(Sx,Px)}{1+d(Ty,Qy)}\right\}d(Px,Qy)$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha < 1$ and $\alpha + \beta < 1$. Then P, Q, S and T have a unique common fixed point in X.

Corollary 3.3 Let P, Q, S and T be four mappings from a complete metric space (X, d) into itself satisfying the conditions (3.1), (3.4) , (3.5) and

$$d(Sx,Ty)\leq\left\{\alpha+\beta\frac{d(Sx,Qy)}{1+d(Ty,Qy)}\right\}d(Px,Qy)$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha < 1$ and $\alpha + 2\beta < 1$.Then P, Q, S and T have a unique common fixed point in X.

Now we generalize the theorem 3.2 for positive integer power of mappings

Theorem 3.3 Let P, Q, S and T be a mappings from a complete metric space (X, d) into itself satisfying the condition (3.1) , for some positive integer s, t, p and q following conditions are as follows:

$S^s(X)\subseteq Q^q(X)$ and $T^t(X)\subseteq P^p(X)$

$$d(S^s x,T^t y)\leq\left\{\alpha+\beta\frac{d(S^s x,P^p x)}{1+d(P^p x,Q^q y)}\right\}d(T^t y,Q^q y) \quad (3.6)$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha < 1$ and $\alpha + \beta < 1$.Suppose that S and T are commuting with P and Q respectively. Then P, Q, S and T have a unique common fixed point in X.

Proof: Since S and T are commuting with P and Q ,therefore S^s and T^t are commuting with P^p and Q^q .Thus by theorem 3.2 , there exists a unique point u in X such that $u = P^p u = Q^q u = S^s u = T^t u$.

From this we obtain $Su = S^s(Su) = P^p(Su)$.Therefore Su is a common fixed point of S^s and P^p. Also, Tu is a common fixed point of T^t and Q^q. Let $x = Su$ and $y = Tu$ in (3.6) , then we obtain

$$d(Su,Tu) = d(S^s x,T^t y)$$

$$\leq\left\{\alpha+\beta\frac{d(S^{s}x,P^{p}x)}{1+d(P^{p}x,Q^{q}y)}\right\}d(T^{t}y,Q^{q}y)$$

$$\leq\left\{\alpha+\beta\frac{d(S^{s}x,S^{s}x)}{1+d(S^{s}x,T^{t}y)}\right\}d(T^{t}y,T^{t}y)$$

i.e. $d(Sz,Tu)\leq 0$

so that $Su=Tu$ .Hence Su is a common fixed point of P^{p}, Q^{q}, S^{s} and T^{t}.

Also we obtain $Pu = S^{s}(Pu) = P^{p}(Pu)$.

Therefore Pu is a common fixed point of S^{s} and P^{p}. Also, Qu is a common fixed point of T^{t} and Q^{q}. Let $x = Pu$ and $y = Qu$ in (3.6), then we obtain

$$d(Pu,Qu)\ =d(S^{s}x,T^{t}y)$$

$$\leq\left\{\alpha+\beta\frac{d(S^{s}x,P^{p}x)}{1+d(P^{p}x,Q^{q}y)}\right\}d(T^{t}y,Q^{q}y)$$

$$\leq\left\{\alpha+\beta\frac{d(S^{s}x,S^{s}x)}{1+d(S^{s}x,T^{t}y)}\right\}d(T^{t}y,T^{t}y)$$

i.e. $d(Pz,Qu)\leq 0$. So that $Pu=Qu$.

Hence Pu is a common fixed point of P^{p}, Q^{q}, S^{s} and T^{t}.

By uniqueness u in X, this show that $u=Pu=Qu=Su=Tu$. Therefore, u is unique common fixed point of P, Q, S and T. This completes the proof.

To illustrate our main theorems 3.2, following example is also given.

Example 3.1 Let $X = [\,0, 1]$, with $d(x,y)\ =|x-y|$

$$Px=Qx=x \quad \text{if} \quad x\in[0,1]$$

$$Tx=Sx=\left|\frac{1}{2}-x\right| \quad \text{if} \quad x\in[0,1]$$

Let $x_n\ =\left(\frac{1}{4}-\frac{1}{4^n}\right)$ be a sequence in X converges to $\frac{1}{4}$ as $n\to\infty$. Hence from definition of sequence (3.2) , the subsequences $\{Sx_n\}$,$\{Px_n\}$, $\{Tx_n\}$ and $\{Qx_n\}$ of $\{x_n\}$ also converges to $\frac{1}{4}$ as $n\to\infty$.

Since $SPx_n=S\left(\frac{1}{4}-\frac{1}{4^n}\right)=\frac{1}{2}-\frac{1}{4}+\frac{1}{4^n}=\frac{1}{4}+\frac{1}{4^n}\to\frac{1}{4}$ as $n\to\infty$

and

$$PSx_n=P\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{4^n}\right)=P\left(\frac{1}{4}+\frac{1}{4^n}\right)\to\frac{1}{4} \text{ as } n\to\infty.$$

Therefore $\lim_{n\to\infty} d\left(SPx_n, PSx_n\right) = d\left(\frac{1}{4},\frac{1}{4}\right) = 0.$ Hence the pair (*S*, *P*) is compatible. Similarly we can show that the pair (*T*, *Q*) is compatible. Also Since the $P(X) = Q(X) = [0,1]$ and $S(X) = T(X) = \left[\ 0, \frac{1}{2}\right]$.

Then clearly conditions:

$S(X) \subseteq Q(X),\ T(X) \subseteq P(X)$ are holds and

$d(Sx,Ty) \le \left\{\alpha + \beta \frac{d(Sx,Px)}{1+d(Px,Qy)}\right\} d(Ty,Qy)$,becomes $|y-x| \le \left\{\alpha + \beta \frac{\left|\frac{1}{2}-2x\right|}{1+|x-y|}\right\}\left|\frac{1}{2}-2y\right|$ is holds on

[0,1) and all other conditions of theorem 3.2 are satisfied by *P*,*Q*.*S* and *T* . Also it is clear that $\frac{1}{4}$ is unique common fixed point of *P*, *Q*, *S* and *T.*

IV. CONCLUSION

In this paper we proved common fixed point theorems for three and four compatible mapping in complete metric space by using new rational inequality which is different and new from some earlier inequality given by Fisher [3, 4], Jungck [8] ,Singh and Chauhan [2] and Lohani and Badshah [11].

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