A STUDY ON WEIBULL DISTRIBUTIONS

Indhumathy.D[1], Seshaiah.C.V[2],Sukkiramathi.K[3]

[1] Assistant Professor, [2] Professor, [3] Assistant Professor,
Department of Mathematics ,Sri Ramakrishna Engineering College, Tamilnadu ,(India)

ABSTRACT

Wind power can be harnessed to provide power for useful task such as generating electricity. Wind is an abundant resource, available in nature that could be utilized by mechanically converting wind power into electricity using wind turbines. The wind power depends entirely on the site and location. This reveals that installing wind turbines, should be done based on the site as well as on the wind speed. So identifying wind speed is very important. The frequency distribution is used to identify the suitable site for wind turbine. The Weibull distribution provides a close approximation to the probability laws of natural phenomena. Weibull distribution has several parametric form and some of their characteristics are revealed in this study.

Keywords-- ***Weibull Distribution, Lognormal Weibull Mixture, Gamma Weibull Mixture , Mixtures Of Weibull Tail***

I. INTRODUCTION

Wind turbines are often placed near the top of hills and ridges and well away from buildings and other structures Wind speed varies with time over several orders of magnitude . Rapid fluctuations in wind speed are called turbulence and may increase the structural and dynamic stresses on the wind turbine components. Thus it is not desirable to install turbines in areas of high turbulence .Near the earths surface wind speed is reduced by friction. Wind turbines operate in the earth's boundary layer. The higher the wind turbine tower, the greater the annual average wind speed. So it is the wind speed distribution which gives the result of wind energy generated. Owing to cost measures we are in a situation to fit a proper distribution to the wind measure. Nowadays Weibull distribution plays an important role in wind speed distribution.

II. WEIBULL DISTRIBUTION

The Weibull distribution is characterized by two parameters, one is the dimensionless shape parameter k and the other is the scale parameter c. The cumulative distribution function and the probability density function are given by equations (1) and (2) respectively as

$$F(v) = 1 - \exp\left[-\left(\frac{v}{c}\right)^k\right] \quad (1)$$

$$f(v) = \frac{dF(v)}{dv} = \frac{k}{c}\left(\frac{k}{c}\right)^{k-1} exp\left[-\left(\frac{v}{c}\right)^k\right] \quad (2)$$

The average wind speed can be expressed as

$$\bar{v} = \int_0^\infty v f(v)\, dv = \int_0^\infty \frac{vk}{c}\left[\left(\frac{v}{c}\right)^{k-1}\right] exp\left[-\left(\frac{v}{c}\right)^{k}\right] dv \qquad (3)$$

Let $x = \left(\frac{v}{c}\right)^k$, $dx = \frac{k}{c}\left(\frac{k}{c}\right)^{k-1} dv$

Sub in (3) we get,

$$\bar{v} = c\int_0^\infty x^{\frac{1}{k}} \exp(-x)\, dx$$

$$\Rightarrow \quad \bar{v} = c\Gamma\left(1+\frac{1}{k}\right) \qquad (4)$$

The standard deviation of wind speed is given by

$$\sigma = \sqrt{\int_0^\infty (v-\bar{v})^2 f(v)\, dv}$$

$$= \sqrt{\int_0^\infty (v^2 - 2v\bar{v} + \bar{v}^2) f(v)\, dv}$$

$$= \sqrt{\int_0^\infty v^2 f(v)\, dv - 2\bar{v}\int_0^\infty v f(v)\, dv + \bar{v}^2}$$

$$= \sqrt{\int_0^\infty v^2 f(v)\, dv - 2\bar{v}\bar{v} + \bar{v}^2} \qquad (5)$$

Using (3) and (4) in equation (5) we get standard deviation in terms of k and c.

III. DOMAIN OF ATTRACTION OF WEIBULL DISTRIBUTION

3.1 Max stable distribution:

A non-degenerate probability distribution μ on R is called maxstable if for a sequence of i.i.d random variables $(X_i)_{i\in N}$ with distribution μ and for each $n \in N$, there exists $a_n > 0$ and $b_n \in R$ such that $\frac{M_n - b_n}{a_n}$ $(M_n = \max_{1\le i\le n} X_i)$ also has distribution function $F(x) = \mu(-\infty, x]$ satisfies for each $n \in N$, there exist $a_n > 0$ and $b_n \in R$ such that

$$F^n(a_n x + b_n) = F(x) \qquad ,\forall \quad x \in R \qquad (6)$$

3.2 Domain of attraction

A distribution μ is said to be in the domain of attraction of a Weibull distribution v, denoted by $\mu \in D(v)$, if there exist $a_n > 0$ and $b_n \in R$ such that the distribution of $\frac{M_n - b_n}{a_n}$ converges weakly to v , where $(M_n = \max_{1\le i\le n} X_i)$ for an i.i.d sequence $(X_i)_{i\in N}$ with distribution μ

3.3 Inverse Weibull distribution

The inverse Weibull distribution is given by

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^{\alpha}} \quad , x > 0\,,\ \alpha > 0\,,\ \theta > 0 \tag{7}$$

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables with cumulative distribution function F(x) and probability density function f(x) . The sequence $\{Z_n{}^{(k)}, n \geq 1\}$ with $Z_n{}^{(k)} = X_{k:L_k(n)+k-1,}\ n = 1,2, \ldots,$ is called the sequence of kth lower record values of $\{X_n, n \geq 1\}$.The pdf of $Z_n{}^{(k)}$ is

$$f_{Z_n{}^{(k)}} == \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\ , \qquad n \geq 1 \tag{8}$$

Proposition :

Let n_0 be any fixed non negative integer ,$-\infty < a < b < \infty, and\ g(x) > 0$ be an absolutely continuous function with $g^{'}(x) \neq 0\ a.e\ on\ (a,b)$. Then the sequence of functions $\{(g(x))^n e^{-g(x)}, n \geq n_0\}$ is complete in $L(a,b)$ iff $g(x)$ is strictly monotone on (a,b).

Proof :

For the inverse Weibull distribution we have $xf(x) = \alpha F(x)\left(-lnF(x)\right)$.

We start with recurrence relations for moments of the inverse Weibull distribution.

Theorem 1: Fix a positive integer $k \geq 1$. Then for any positive integer r, we have

$$E\left(Z_n{}^{(k)}\right)^r = \left(1 - \frac{r}{(n-1)\alpha}\right) E\left(Z_{n-1}{}^{(k)}\right)^r$$

$$= \prod_{i=1}^{n-1}\left(1 - \frac{r}{i\alpha}\right) E\left(X_{k:k}\right)^r \tag{9}$$

Proof :

For $n \geq 1\ and\ r = 1,2, \ldots,$ from () we have

$$E\left(Z_n{}^{(k)}\right)^r = \frac{\alpha k^n}{(n-1)!}\int x^{r-1}[-\ln F(x)]^n [F(x)]^k\, dx. \tag{10}$$

Integrating the above integral by parts (10), we get

$$E\left(Z_{n-1}{}^{(k)}\right)^r = \frac{\alpha(n-1)}{r}\left[E\left(Z_{n-1}{}^{(k)}\right)^r - E\left(Z_n{}^{(k)}\right)^r\right]$$

$$\Rightarrow \qquad E\left(Z_n{}^{(k)}\right)^r = \left(1-\frac{r}{(n-1)\alpha}\right)E\left(Z_{n-1}{}^{(k)}\right)^r \qquad (11)$$

Using induction we get (9).

3.4 Weibull Conditional survival function

Conditional properties:

Suppose $Pr(X > x/Y > y) = \exp\{[-x/\sigma_1(y)]^{\gamma_1}\}, \qquad x, y > 0$

And $Pr(Y > y/X > x) = \exp\{[-y/\sigma_2]^{\gamma_2}\}, \qquad x, y > 0$

Expression of the joint survival function

$H(x,y) = exp\left\{-\left[\left(\frac{x}{\sigma_1}\right)^{\gamma_1} + \left(\frac{y}{\sigma_2}\right)^{\gamma_2} + \theta\left(\frac{x}{\sigma_1}\right)^{\gamma_1}\left(\frac{y}{\theta_2}\right)^{\gamma_2}\right]\right\}, \quad x, y > 0$ where $\sigma_1, \sigma_2 > 0 \quad and \ 0 \leq \theta$.

3.5 Domain of attraction of Weibull distribution

Let $\alpha > 0$, and let $\emptyset_\alpha$ be the distribution function of a Weibull distribution with index α, ie., $\emptyset_\alpha(x) = e^{-(-x)^\alpha}$ if $x < 0, \emptyset_\alpha(x) = 1 \ if \ x \geq 0$. Then a distribution function F is in the domain of attraction of $\emptyset_\alpha$ iff there exists $x_0 \in R$, such that $\mu(-\infty, x_0) = 1$ and $\mu(x_0 - \epsilon, x_0) = e^\alpha L(1/\epsilon)$ for some slowly varying function L as $\epsilon \longrightarrow 0$. In this case, we can set $\gamma_n = \left(\frac{1}{1-F}\right)(n)$, and then

$$F^n(x_0 + (x_0 - \gamma_n)x) \underset{n\to\infty}{\to} \emptyset_\alpha \quad , \forall x \in R.$$

IV. OTHER TYPES OF WEIBULL DISTRIBUTIONS

4.1 Three parameter Weibull distribution

The Weibull probability distribution function, which is a three parameter function, can be expressed mathematically as

$$f(v) = \frac{k}{c-\varepsilon}\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} exp\left[-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right]$$

Where v is the wind speed, k is the non-dimensional shape parameter, c is the scale parameter and ε is the location parameter.

Its cumulative distribution function can be written as

$$F(v) = 1 - exp\left[-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right]$$

4.2 Modified Weibull distribution

The PDF and CDF of Modified Weibull distribution are given respectively by $f(x) = f(x;\mu,\delta,\gamma) = \mu(\delta+\gamma x)x^{\delta-1}exp\{\gamma x - \mu x^{\delta}e^{\gamma x}\}$, $x > 0$ where $\mu > 0, \delta \geq 0$ and $\gamma > 0$ are parameters namely scale, shape and accelerating respectively.

4.3 Log Modified Weibull distribution:

Let X have PDF (1) and define $Y = \log X$, then Y is said to have a Log Modified Weibull distribution with parameters $\mu > 0, \delta \geq 0$ and $\gamma > 0$. The PDF and CDF of Y are given, respectively by

$$f(y) = \mu(\delta + \gamma e^{y})exp\{\delta y + \gamma e^{y} - \mu e^{\delta y + \gamma e^{y}}\}$$

$$F(y) = 1 - exp\{-\mu e^{\delta y + \gamma e^{y}}\}, y \in R.$$

4.4 Bivariate Weibull distribution

The Bivariate Weibull distribution is given by $\bar{F}(x,y) = exp\left\{-\left[\left(\frac{x}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{y}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{\alpha}\right\}$ where $0 < \alpha \leq 1, 0 < \lambda_1, \lambda_2 < \infty$ and $0 < \gamma_1, \gamma_2 < \infty$.

Lu and Bhattacharya (1990) developed a joint survival function by letting $h_1(x)$ and $h_2(y)$ be two arbitrary failure rate functions on $[0,\infty)$ and $H_1(x)$ and $H_2(y)$ be their corresponding cumulative failure rate. Given the stress $S = s > 0$, th joint survival function conditioned on S, as they are defined as $\bar{F}(x,y/s) = exp\{-[H_1(x) + H_2(y)]^{\gamma}s\}$ where γ measures the conditional association of X and Y.

V. MIXTURES OF WEIBULL DISTRIBUTION

5.1 Two component mixture Weibull distribution:

The probability density function, which depends on five parameters $(v; k_1, c_1, k_2, c_2, w)$ is given by

$$ff(v;k_1,c_1,k_2,c_2,w) = w\, f(v;k_1,c_1) + (1-w)f(v;k_2,c_2)$$

The cumulative distribution function is given by

$$FF(v;k_1,c_1,k_2,c_2,w) = w\, F(v;k_1,c_1) + (1-w)F(v;k_2,c_2)$$

5.2 Mixture Gamma and Weibull distribution

The probability density function and cumulative distribution function of the mixture gamma and Weibull distribution are given by

$$h(v;\alpha,\beta,k,c,w) = wg(v;\alpha,\beta) + (1-w)f(v;k,c)$$

$H(v;\alpha,\beta,k,c,w) = wG(v;\alpha,\beta) + (1-w)F(v;k,c)$

5.3 Mixture Normal and Weibull distribution

The probability density function of the mixture Normal and Weibull distribution comprising of truncated normal and conventional Weibull distribution is written as

$s(v;\mu,\sigma,k,c) = wq(v;\mu,\sigma) + (1-w)f(v;k,c)$

Its cumulative distribution function is given as

$S(v;\mu,\sigma,k,c) = wQ(v;\mu,\sigma) + (1-w)F(v;k,c)$

5.4 Mixture Weibull and GEV distribution

The probability density function of the mixture distribution comprising Weibull and GEV functions which is written as

$t(v;k,c,\zeta,\delta,l) = wf(v;k,c) + (1-w)e(v;\zeta,\delta,l)$

Its cumulative distribution function is given by

$T(v;k,c,\zeta,\delta,l) = wF(v;k,c) + (1-w)E(v;k,c,\zeta,\delta,l)$

5.5 Mixture Weibull and lognormal distribution

The probability density function of the mixture distribution comprising Weibull and lognormal functions which is written as

$u(v;k,c,\lambda,\varphi) = wf(v;k,c) + (1-w)l(v;\ \lambda,\varphi)$

Its cumulative distribution function is given by

$U(v;k,c,\lambda,\varphi) = wF(v;k,c) + (1-w)L(v;\ \lambda,\varphi)$

5.6 Mixtures of Weibull tail distribution

Weibull tail-distributions are defined through their distribution function by

$F(x) = 1 - exp\{-x^{1/\theta}\, l(x)\}$, where l is a slowly-varying function.

5.6.1 Theorem

Let F be defined by $F(x) = 1 - p\ exp\{-x^{1/\theta_1} l_1(x)\} + (p-1)exp\{-x^{1/\theta_2} l_2(x)\}$. Then F is a Weibull tail-distribution with associated Weibull tail-coefficient θ_2.

Proof : $1 - F(x) = p\,exp\{-x^{1/\theta_1} l_1(x)\} + (1-p)exp\{-x^{1/\theta_2} l_2(x)\}$

$$= exp\{-x^{1/\theta_2} l_2(x)\}(1 - p + p\,\exp\{-x^{1/\theta_1} l_1(x) + x^{1/\theta_2} l_2(x)\})$$

$$= exp\{-x^{1/\theta_2} l_2(x)\}(1 - p + p\,\exp\{-x^{1/\theta_1} l_1(x)(1 - x^{1/\theta_2 - 1/\theta_1} l_2(x)/l_1(x)\})$$

$$= exp\{-x^{1/\theta_2} l_2(x)\}(1 - p + p\,\exp\{-x^{1/\theta_1} l_1(x)(1 + 0(1)\})$$

$$= exp\{-x^{1/\theta_2} l_2(x)\}exp\{-x^{1/\theta_2} l_2(x)\}(1 - p + p(1 + 0(1)))$$

$$= exp\{-x^{1/\theta_2} l_2(x)\}1 + 0(1) \quad (12)$$

Which (12) is again a Weibull tail-distribution with coefficient θ_1 and θ_2.

5.6.2 Theorem

The Weibull distribution has the scaling property . That is $X \sim W(\alpha, \beta)$ then $Y = kX$ also has Weibull distribution.

Proof :

Let the Random Variable X has the $Weibull\,(\alpha, \beta)$ distribution with probability density function

$$f(x) = \frac{\beta}{\alpha} x^{\beta-1} e^{-(1/\alpha)x^{\beta}}\ , x > 0. \quad (13)$$

Let k be a positive , real constant.

The transformation $Y = g(X) = k(X)$ is a $1-1$ transformation from $\chi = \{x/x > 0\}$ to $Y = \{y/y > 0\}$ with inverse $X = g^{-1}(Y) = Y/k$ and Jacobian $\frac{dX}{dY} = \frac{1}{k}$. Therefore , by the transformation technique, the pdf of Y is

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$= \frac{\beta}{\alpha}(y/k)^{\beta-1} e^{-(1/\alpha)(y/k)^{\beta}}[1/k]$$

$$= \frac{\beta}{\alpha k^{\beta}} y^{\beta-1} e^{(-1/\alpha k^{\beta})y^{\beta}} \quad , y > 0$$

Which (13) is the pdf of $W(\alpha k^{\beta}, \beta)$ Random Variable.

VI. CONCLUSION

Wind power stochastic characteristics play a significant role in planning, design and operation of the wind turbines. It is therefore necessary to develop analytical relationships between wind speed characteristics which are functions of recorded wind speed behavior and the wind speed statistics. In this study we had a review of the Weibull distributions ,its modifications, Mixtures of Weibull with the already existing probability distributions. These distributions are used to estimate the wind speed at several locations with help of Weibull parameters.

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