

A COUPLED FIXED POINT THEOREM IN FUZZY METRIC SPACE

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ABSTRACT

The purpose of this paper is to introduce a coupled fixed point theorem and t- Norm of Hadžić type in fuzzy metric space. We also give an application of our result.

Keywords: *Coupled Fixed Points, Fuzzy Metric Space, Occasionally Weakly Compatible Maps*

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I. INTRODUCTION

The evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [10] in 1965, as a new way to represent the vagueness in everyday life. In mathematical programming, problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution, that is quite useful, is the one using fuzzy sets. Zadeh's [10] investigation of the notion of fuzzy set has led to a rich growth of fuzzy Mathematics. Many authors have introduced the concept of fuzzy metric in different ways. There are many view points of the notion of the fuzzy metric spaces in fuzzy topology (see Deng [1], Erceg [2], George and Veeramani [3], Kaleva and Seikkala [5], Kramosil and Michálek [6]). In 2009, Lakshmikantham et. al. [7] introduced coupled fixed point theorems in partially ordered metric spaces. Motivated by them, we introduce a coupled fixed point theorem and t-Norm of Hadžić type in fuzzy metric space.

II. PRELIMINARIES

Definition 2.1 A t-norm t is a 2-place function $t : [0,1] \times [0, 1] \rightarrow [0,1]$ which satisfies the following conditions :

for all $a, b, c, d \in [0,1]$.

- (i) $t(0, 0) = 0$;
- (ii) $t(0, 1) = 1$;
- (iii) $t(a, b) = t(b, a)$;
- (iv) if $a \leq c$ and $b \leq d$, then $t(a, b) \leq t(c, d)$;
- (v) $t(t(a, b), c) = t(a, t(b, c))$.

Definition 2.2 Let $\sup_{0 < t < 1} \Delta(t, t) = 1$. A t-norm Δ is said to be of Hadžić-type

(H-type) if the family of functions $\left\{ \Delta^m(t) \right\}_{m=1}^{\infty}$ is equi-continuous at $t = 1$, where

$$\Delta^1(t) = t, \Delta^{m+1}(t) = \Delta(\Delta^m(t)), m = 1, 2, 3, \dots, t \in [0, 1]$$

The t-norm $\Delta_M = \min$ is an example of H-type.

Remark 2.1 Δ is a H-type if and only if for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that following implications hold :

$$t > 1 - \delta(\lambda) \text{ implies } \Delta^m(t) > 1 - \lambda.$$

Remark 2.2 [4] Every t-norm Δ_M is of Hadžić-type but converse need not be true.

Definition 2.3 [6] The 3-tuple (X, M, t) is called fuzzy metric space if X is an arbitrary set, t is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$

- (i) $M(x, y, 0) = 0$;
- (ii) $M(x, y, t) = 1$ if and only if $x = y$;
- (iii) $M(x, y, t) = M(y, x, t)$;
- (iv) $M(x, z, t+s) \geq t(M(x, y, t), M(y, z, s))$;
- (v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

Note that the function $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t .

In our theory, we consider the following conditions :

- (vi) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all x, y in X .

Definition 2.4 A sequence $\{x_n\}$ in a fuzzy metric space (X, M, t) is said to be

- (i) convergent to x in X if for every $\varepsilon > 0$ and $\lambda > 0$ there is an integer $n_0 \in \mathbb{N}$ such that $M(x_n, x, \varepsilon) > 1 - \lambda$, for all $n \geq n_0$.
- (ii) Cauchy sequence in X if for every $\varepsilon > 0$ and $\lambda > 0$ there is an integer $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$, for all $m, n \geq n_0$.
- (iii) complete if every Cauchy sequence in X is convergent in X .

Definition 2.5 Define $\Phi = \{\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\}$, where $\mathbb{R}^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfying the following conditions :

- (ϕ_1) ϕ is non-decreasing.
- (ϕ_2) ϕ is upper semi-continuous from the right.
- (ϕ_3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in \mathbb{N}$.

Clearly, if $\phi \in \Phi$ then $\phi(t) < t$ for all $t > 0$.

Definition 2.6 An element $x \in X$ is called the common fixed point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $x = f(x, x) = g(x)$.

Definition 2.7 An element $(x, y) \in X \times X$ is called a

- (i) coupled fixed point of the mapping $f : X \times X \rightarrow X$ if $f(x, y) = x$, $f(y, x) = y$.

- (ii) coupled coincidence point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $f(x, y) = g(x)$, $f(y, x) = g(y)$.
- (iii) common coupled fixed point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $x = f(x, y) = g(x)$, $y = f(y, x) = g(y)$.

Definition 2.8 The mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ are called commutative if

$$gf(x, y) = f(gx, gy) \text{ for all } x, y \in X.$$

Definition 2.9 The mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ are called occasionally weakly compatible if there is a point $x \in X$ which is a coupled coincidence point of f and g at which f and g both commute.

Remark 2.3 For convenience, we denote

$$(2.1) [M(x, y, t)]^n = \underbrace{M(x, y, t) * M(x, y, t) * \dots * M(x, y, t)}_n, \text{ for all } n \in \mathbb{N}.$$

III. MAIN RESULT.

Theorem 3.1 Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm of H-type. Suppose $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings and there exists $\phi \in \Phi$ such that

$$(3.1) M(f(x, y), f(u, v), \phi(t)) \geq \psi[M(gx, gu, t) * M(gy, gv, t)],$$

for all $x, y, u, v \in X$ and $t > 0$, where $\psi : [0, 1] \rightarrow [0, 1]$ is continuous function such that $\psi(t) \geq t$ for all $t \in [0, 1]$.

Suppose that $f(X \times X) \subseteq g(X)$, f and g are occasionally weakly compatible, range space of one of the mappings f or g is complete. Then f and g have a coupled coincidence point. Moreover, there exists a unique point x in X such that $x = f(x, x) = g(x)$.

Proof. Let x_0, y_0 be two arbitrary points in X . Since $f(X \times X) \subseteq g(X)$, we can choose x_1, y_1 in X such that $g(x_1) = f(x_0, y_0)$, $g(y_1) = f(y_0, x_0)$.

Proceeding in this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1}) = f(x_n, y_n)$ and $g(y_{n+1}) = f(y_n, x_n)$ for all $n \geq 0$.

Step I. We first show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(3.2) \underbrace{(1 - \delta) * (1 - \delta) * \dots * (1 - \delta)}_p \geq (1 - \varepsilon), \text{ for all } p \in \mathbb{N}.$$

Since $\lim_{t \rightarrow \infty} M(x, y, t) = 1$, for all x, y in X , there exists $t_0 > 0$ such that

$$M(gx_0, gx_1, t_0) \geq (1 - \delta) \text{ and } M(gy_0, gy_1, t_0) \geq (1 - \delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3) , we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$(3.3) t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$M(gx_1, gx_2, \phi(t_0)) = M(f(x_0, y_0), f(x_1, y_1), \phi(t_0))$$

$$\begin{aligned}
&\geq \psi[M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0)] \\
&\geq M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0) \\
M(gy_1, gy_2, \phi(t_0)) &= M(f(y_0, x_0), f(y_1, x_1), \phi(t_0)) \\
&\geq \psi[M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0)] \\
&\geq M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0).
\end{aligned}$$

Similarly, we can also get

$$\begin{aligned}
M(gx_2, gx_3, \phi^2(t_0)) &= M(f(x_1, y_1), f(x_2, y_2), \phi^2(t_0)) \\
&\geq \psi[M(gx_1, gx_2, t_0) * M(gy_1, gy_2, t_0)] \\
&\geq [M(gx_0, gx_1, t_0)]^2 * [M(gy_0, gy_1, t_0)]^2 \\
M(gy_2, gy_3, \phi^2(t_0)) &= M(f(y_1, x_1), f(y_2, x_2), \phi^2(t_0)) \\
&\geq [M(gy_0, gy_1, t_0)]^2 * [M(gx_0, gx_1, t_0)]^2.
\end{aligned}$$

Continuing in this way, we can get

$$\begin{aligned}
M(gx_n, gx_{n+1}, \phi^n(t_0)) &\geq [M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}} \\
M(gy_n, gy_{n+1}, \phi^n(t_0)) &\geq [M(gy_0, gy_1, t_0)]^{2^{n-1}} * [M(gx_0, gx_1, t_0)]^{2^{n-1}}.
\end{aligned}$$

So, from (3.2) and (3.3), for $m > n \geq n_0$, we have

$$\begin{aligned}
M(gx_n, gx_m, t) &\geq M\left(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\
&\geq M\left(gx_n, gx_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) \\
&\geq M(gx_n, gx_{n+1}, \phi^n(t_0)) * M(gx_{n+1}, gx_{n+2}, \phi^{n+1}(t_0)) \\
&\quad * \dots * M(gx_{m-1}, gx_m, \phi^{m-1}(t_0)) \\
&\geq \{[M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}}\} \\
&\quad * \{[M(gx_0, gx_1, t_0)]^{2^n} * [M(gy_0, gy_1, t_0)]^{2^n}\} * \dots \\
&\quad * \{[M(gx_0, gx_1, t_0)]^{2^{m-2}} * [M(gy_0, gy_1, t_0)]^{2^{m-2}}\} \\
&= [M(gx_0, gx_1, t_0)]^{2^{n-1}(2^{m-n-1})} * [M(gy_0, gy_1, t_0)]^{2^{n-1}(2^{m-n-1})} \\
&\geq \underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_{2^{n-1}(2^{m-n-1})} \geq (1-\varepsilon),
\end{aligned}$$

which implies that

$$M(gx_n, gx_m, t) \geq (1 - \varepsilon) \text{ for all } m, n \in \mathbb{N} \text{ with } m > n \geq n_0 \text{ and } t > 0.$$

Therefore, $\{gx_n\}$ is a Cauchy sequence in X .

Similarly, $\{gy_n\}$ is also a Cauchy sequence in X .

Step II. Now, we show that f and g have coupled coincidence point. Without loss of generality, we can assume that $g(X)$ is complete, there exist points x, y in $g(X)$ so that $\lim_{n \rightarrow \infty} g(x_{n+1}) = x$ and $\lim_{n \rightarrow \infty} g(y_{n+1}) = y$.

For $x, y \in g(X)$ implies the existence of p, q in X such that $g(p) = x, g(q) = y$ and hence

$$\begin{aligned}
\lim_{n \rightarrow \infty} g(x_{n+1}) &= \lim_{n \rightarrow \infty} f(x_n, y_n) = g(p) = x, \\
\lim_{n \rightarrow \infty} g(y_{n+1}) &= \lim_{n \rightarrow \infty} f(y_n, x_n) = g(q) = y.
\end{aligned}$$

From (3.1), we have

$$\begin{aligned} M(f(x_n, y_n), f(p, q), \phi(t)) &\geq \psi[M(gx_n, g(p), t) * M(gy_n, g(q), t)] \\ &\geq [M(gx_n, g(p), t) * M(gy_n, g(q), t)]. \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we have

$$M(g(p), f(p, q), \phi(t)) = 1, \text{ i.e., } f(p, q) = g(p) = x.$$

Similarly, $f(q, p) = g(q) = y$.

Since, f and g are occasionally weakly compatible, so $gf(p, q) = f(g(p), g(q))$ and $gf(q, p) = f(g(q), g(p))$,

i.e., $g(x) = f(x, y)$ and $g(y) = f(y, x)$.

Thus, f and g have a coupled coincidence point.

Step III. Now we show that $g(x) = x$ and $g(y) = y$.

Since $*$ is a t -norm of H-type, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_p \geq (1-\varepsilon), \text{ for all } p \in \mathbb{N}.$$

Also, since $\phi \in \Phi$, using condition (ϕ_3) , we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(gx, x, \phi(t_0)) &= M(f(x, y), f(p, q), \phi(t_0)) \\ &\geq \psi[M(gx, gp, t_0) * M(gy, gq, t_0)] \\ &\geq M(gx, gp, t_0) * M(gy, gq, t_0). \end{aligned}$$

Similarly,

$$M(gy, y, \phi(t_0)) \geq M(gy, gq, t_0) * M(gx, gp, t_0).$$

Continuing in the same way, for all $n \in \mathbb{N}$

$$M(gx, x, \phi^n(t_0)) \geq [M(gx, gx, t_0)]^{2^{n-1}} * [M(gy, y, t_0)]^{2^{n-1}}$$

Thus, we have

$$\begin{aligned} M(gx, x, t) &\geq M\left(gx, x, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(gx, x, \phi^{n_0}(t_0)\right) \\ &\geq [M(gx, x, t_0)]^{2^{n_0-1}} * [M(gy, y, t_0)]^{2^{n_0-1}} \\ &\underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \text{ for all } n_0 \in \mathbb{N}. \end{aligned}$$

So, for any $\varepsilon > 0$, we have

$$M(gx, y, t) \geq (1 - \varepsilon), \text{ for all } t > 0.$$

This implies that $g(x) = x$. Similarly, $g(y) = y$.

Step IV. Now we show that $x = y$.

Since $*$ is a t-norm of H-type for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_p \geq (1-\varepsilon), \text{ for all } p \in \mathbb{N}.$$

Since $\lim_{t \rightarrow \infty} M(x, y, t) = 1$, for all x, y in X , there exists $t_0 > 0$ such that

$$M(x, y, t_0) \geq (1 - \delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3) , we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(x, y, \phi(t_0)) &= M(f(p, q), f(q, p), \phi(t_0)) \\ &\geq \psi[M(gp, gq, t_0) * M(gq, gp, t_0)] \\ &\geq M(gp, gq, t_0) * M(gq, gp, t_0) \\ &= [M(x, y, t_0)]^2. \end{aligned}$$

Continuing likewise, we have for all $n \in \mathbb{N}$, that

$$M(x, y, \phi^n(t_0)) \geq [M(x, y, t_0)]^{2^n}.$$

Thus, we have

$$\begin{aligned} M(x, y, t) &\geq M\left(x, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M(x, y, \phi^{n_0}(t_0)) \\ &\geq [M(x, y, t_0)]^{2^{n_0}}. \end{aligned}$$

Thus, we have

$$\geq \underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \text{ which implies that } x = y. \text{ Thus, } f \text{ and } g \text{ have common fixed}$$

point x in X .

Step V. Uniqueness :

Let z be any point in X such that $z \neq x$ with $g(z) = z = f(z, z)$.

Since $*$ is a t-norm of H-type for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_p \geq (1-\varepsilon), \text{ for all } p \in \mathbb{N}.$$

Since $\lim_{t \rightarrow \infty} M(x, y, t) = 1$, for all x, y in X , there exists $t_0 > 0$ such that

$$M(x, y, t_0) \geq (1 - \delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3) , we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(x, z, \phi(t_0)) &= M(f(x, x), f(z, z), \phi(t_0)) \\ &\geq \psi[M(g(x), g(z), t_0) * M(g(x), g(z), t_0)] \\ &\geq M(g(x), g(z), (t_0)) * M(g(x), g(z), (t_0)) \\ &= [M(x, z, t_0)]^2. \end{aligned}$$

Thus, we have $M(x, z, t) \geq M\left(x, z, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right)$

$$\begin{aligned} &\geq M\left(x, z, \phi^{n_0}(t_0)\right) \\ &\geq ([M(x, z, t_0)]^{2^{n_0-1}})^2 \\ &\geq (M(x, z, t_0))^{2^{n_0}} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \dots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \end{aligned}$$

which implies that $x = z$. Thus, f and g have a unique common fixed point in X .

IV. AN APPLICATION.

Theorem 4.1 Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm define by $a * b = \min\{a, b\}$ for all a, b in X . Suppose A, B be occasionally weakly compatible self maps on X satisfying the following condition

$$(4.1) \quad M(X) \subseteq N(X),$$

$$(4.2) \quad \text{there exists } \phi \in \Phi \text{ such that}$$

$$M(Ax, Ay, \phi(t)) \geq \psi[M(Bx, By, t)], \text{ for all } x, y, \in X \text{ and } t > 0,$$

where $\psi : [0, 1] \rightarrow [0, 1]$ is continuous function such that $\psi(t) \geq t$ for all $t \in [0, 1]$.

If range of any one of the maps A or B is complete, then A and B have a unique common fixed point in X .

Proof. Assuming $f(x, y) = A(x)$ and $g(x) = A(x)$ for all $x, y \in X$ in theorem 3.1, we get the proof.

Taking $\phi(t) = kt$, $k \in (0, 1)$ and $\psi(t) = t$, we have the following :

Corollary 4.1 Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm defined by $a * b = \min\{a, b\}$ for all a, b in X . Suppose A, B be occasionally weakly compatible self maps on X satisfying (4.1) and the following condition :

$$(4.2) \quad \text{there exists } k \in (0, 1) \text{ such that}$$

$$M(Ax, Ay, kt) \geq M(Bx, By, t), \text{ for all } x, y \in X \text{ and } t > 0.$$

If range of any one of the maps A or B is complete, then A and B have a unique common fixed point in X .

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