FIXED POINTS IN MENGER SPACE FOR COMPATIBILITY OF TYPE (β) AND OCCASIONALLY WEAKLY COMPATIBLE

V. K. Gupta[1], Arihant Jain[2], Ramesh Bhinde[3]

[1]*Department of Mathematics, Govt. Madhav Science College, Ujjain M.P. (India)*

[2]*Department of Applied Mathematics,*

Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P. (India)

[3]*Department of Mathematics, Govt. P. G. College, Alirajpur M.P. (India)*

ABSTRACT

The purpose of this paper is to prove a common fixed point theorem for six self maps which generalizes the result of Pant et. al. [8], using the concept of compatibility of type (β) and occasionally weak compatibility in Menger space.

Keywords and Phrases: Menger space, Common fixed points, Compatible maps, Compatible maps of type (β) and weak compatibility

AMS Subject Classification (2000). Primary 47H10, Secondary 54H25

I. INTRODUCTION

In 1942, Professor Karl Menger [7] has introduced the theory of probabilistic metric space in which a distribution function was used instead of non-negative real number as value of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. The development of fixed point theory in PM-spaces was due to Schweizer and Sklar [11]. Sehgal and Bharucha-Reid [12] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Jungck and Rhoades [6] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [13] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [5] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [8].

Cho, Murthy and Stojakovik [1] proposed the concept of compatible maps of type (A) in Menger space and gave some fixed point theorems. Recently, using the concept of compatible mappings of type (A), Jain et. al. [3, 4] proved some interesting fixed point theorems in Menger space. In the sequel, Patel and Patel [10] proved a common fixed point theorem for four compatible maps of type (A) in Menger space by taking a new inequality. Recently, in 2013, Jain et. al. [2] proved a common fixed point theorem using the concept of semi-compatibility and occasionally weak compatibility in Menger space.

In this paper a fixed point theorem for six self maps has been proved using the concept of occasionally weak compatibility and compatibility of type (β) which generalizes the result of Pant et. al. [9]. We also cited an example.

II. PRELIMINARIES

Definition 2.1.[7] A mapping $\mathcal{F} : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with inf $\{ \mathcal{F}(t) \mid t \in R \} = 0$ and $\sup \{ \mathcal{F}(t) \mid t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [2] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions :

(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a$, $d \ge b$,

(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [2] A *probabilistic metric space (PM-space)* is an ordered pair $(X, \mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u, v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$, for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [2] A *Menger space* is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \ge t \{F_{u, v}(x), F_{v, w}(y) \}$, for all $u, v, w \in X$, $x, y \ge 0$.

Definition 2.5. [11] A sequence $\{x_n\}$ in a Menger space $(X, \mathcal{F}, t)$ is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \text{ for all } n \ge M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x_m}(\varepsilon) > 1- \lambda \qquad \text{for all } m, n \ge M(\varepsilon, \lambda).$$

A Menger PM-space (X, $\mathcal{F}$, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way :

Proposition 2.1. [3] If (X, d) is a metric space then the metric d induces mappings $\mathcal{F} : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, p, q $\in$ X, where

$$H(k) = 0, \quad \text{for } k \le 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by t(a,b) = min {a, b}. Then (X, $\mathcal{F}$, t) is a Menger space. It is complete if (X, d) is complete.

The space (X, $\mathcal{F}$, t) so obtained is called the *induced Menger space.*

Definition 2.6. [6] Self mappings A and S of a Menger space (X, $\mathcal{F}$, t) are said to be weak compatible if they commute at their coincidence points i.e. Ax = Sx for x $\in$ X implies ASx = SAx.

Definition 2.7. [8] Self mappings A and S of a Menger space (X, **F**, t) are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all x > 0, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [1] Self maps S and T of a Menger space (X, $\mathcal{F}$, t) are said to be *compatible of type (β)* if $F_{SSx_n, TTx_n}(x) \to 1$ for all x > 0, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.9. [9] Self maps S and T of a Menger space (X, $\mathcal{F}$, t) are said to be *semi-compatible* if $F_{STx_n, Tu}(x) \to 1$ for all x > 0, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.10. [2] Self maps A and S of a Menger space (X, $\mathcal{F}$, t) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Remark 2.1. [2] The concept of occasionally weakly compatible is more general than that of weak compatibility.

Now, we give an example of pair of self maps (I, L) which are compatible of type (β) but not-semi-compatible.

Example 2.1. Let (X, d) be a metric space where X = [0, 1] and (X, $\mathcal{F}$, t) be the induced Menger space with

$$F_{x,y} = \frac{t}{t + d(x, y)} \quad \text{for all } t > 0.$$

Define self maps I and L as follows :

$$I(x) = x \text{ for all } x \in X \quad \text{and} \quad L(x) = \begin{cases} x, & \text{if } 0 \le x < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Taking $x_n = \frac{1}{2} - \frac{1}{n}$, we get $Ix_n = x_n = \frac{1}{2} - \frac{1}{n}$ and $Lx_n = \frac{1}{2} - \frac{1}{n}$.

Thus, $Lx_n \to \frac{1}{2}$ as $n \to \infty$ and $Ix_n \to \frac{1}{2}$, as $n \to \infty$.

Hence, $x = \frac{1}{2}$

Since $Lx_n = \frac{1}{2} - \frac{1}{n}$

Therefore, $ILx_n = I\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$

and $LLx_n = L\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$.

Consider $\lim_{n\to\infty} F_{ILx_n, LLx_n}(t) = \lim_{n\to\infty} F_{\frac{1}{2}-\frac{1}{n}, \frac{1}{2}-\frac{1}{n}}(t) = 1$ for $t > 0$.

Also, $LIx_n = L\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$ and $IIx_n = I\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$.

Consider $\lim_{n\to\infty} F_{LLx_n, IIx_n}(t) = \lim_{n\to\infty} F_{\frac{1}{2}-\frac{1}{n}, \frac{1}{2}-\frac{1}{n}}(t) = 1$ for $t > 0$.

Therefore, by definition, (I, L) is compatible mapping of type (A).

Now, $\lim_{n\to\infty} F_{ILx_n, Lx}(t) = \lim_{n\to\infty} F_{\frac{1}{2}-\frac{1}{n}, 1}(t) < 1$ for $t > 0$.

Therefore, (I, L) is not semi-compatible mapping. Thus the pair (I, L) of self maps is compatible of type (β) but not semi-compatible.

Remark 2.2. In view of above example, it follows that the concept of compatible maps of type (β) is more general than that of semi-compatible maps.

Lemma 2.1. [15] Let $\{x_n\}$ be a sequence in a Menger space (X, $\mathcal{F}$, t) with continuous t-norms t and t(a, a) ≥ a. If there exists a constant $k \in (0, 1)$ such that $F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t)$ for all $t \geq 0$ and n = 1, 2, 3, ..., then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.3. [15] Let (X, $\mathcal{F}$, t) be a Menger space. If there exists a constant $k \in (0, 1)$ such that

$$F_{x, y}(kt) \geq F_{x, y}(t) \text{ for all } x, y \in X \text{ and } t > 0, \text{ then } x = y.$$

A class of implicit relation. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \to R$, non-decreasing in the first argument with the property :

a. For u, v ≥ 0, $\phi(u, v, v, u) \geq 0$ or $\phi(u,v,u,v) \geq 0$ implies that u ≥ v.

b. $\phi(u, u, 1, 1) \geq 0$ implies that u ≥ 1.

Example 2.3. Define $\phi(t_1,t_2,t_3,t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

III. MAIN RESULT

Theorem 3.1. Let A, B, L, M, S and T be self mappings on a complete Menger space (X, $\mathcal{F}$, t) with t(a, a) ≥ a, for some a ∈ [0, 1], satisfying :

(3.1.1) L(X) ⊆ ST(X), M(X) ⊆ AB(X);

(3.1.2) ST(X) and AB(X) are complete subspace of X;

(3.1.3) either AB or L is continuous;

(3.1.4) (L, AB) is compatible maps of type (β) and (M, ST) is occasionally weak compatible;

(3.1.5) for some φ ∈ Φ, there exists k ∈ (0, 1) such that for all x, y ∈ X and t > 0,

$$\phi(F_{Lx, My}(kt), F_{ABx, STy}(t), F_{Lx, ABx}(t), F_{My, STy}(kt)) \geq 0$$

then A, B, L, M, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From condition (3.1.1) $\exists$ $x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \quad \text{and} \quad Mx_1 = ABx_2 = y_1.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \quad \text{and} \quad Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$$

for n = 0, 1, 2, ...

Step 1. Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lx_{2n},\, Mx_{2n+1}}(kt),\, F_{ABx_{2n},\, STx_{2n+1}}(t),\, F_{Lx_{2n},\, ABx_{2n}}(t),\, F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{y_{2n},\, y_{2n+1}}(kt),\, F_{y_{2n-1},\, y_{2n}}(t),\, F_{y_{2n},\, y_{2n-1}}(t),\, F_{y_{2n+1},\, y_{2n}}(kt)) \geq 0.$$

Using (a), we get

$$F_{y_{2n},\, y_{2n+1}}(kt) \geq F_{y_{2n-1},\, y_{2n}}(t).$$

Therefore, for all n even or odd, we have

$$F_{y_n,\, y_{n+1}}(kt) \geq F_{y_{n-1},\, y_n}(t).$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X, which is complete.

Hence $\{y_n\} \to z \in X$. Also its subsequences converges as follows :

$\{Lx_{2n}\} \to z$, $\{ABx_{2n}\} \to z$, $\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$.

Case I. When AB is continuous.

As AB is continuous, $(AB)^2x_{2n} \to ABz$ and $(AB)Lx_{2n} \to ABz$.

As (L, AB) is compatible pair of type (β), so

$LLx_{2n} \to (AB)(AB)x_{2n}$ and so $LABx_{2n} \to ABz$

Step 2. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LABx_{2n},\, Mx_{2n+1}}(kt),\, F_{ABABx_{2n},\, STx_{2n+1}}(t),\, F_{LABx_{2n},\, ABABx_{2n}}(t),\, F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(F_{ABz,\, z}(kt),\, F_{ABz,\, z}(t),\, F_{ABz,\, ABz}(t),\, F_{z,\, z}(kt)) \geq 0$$

$$\phi(F_{ABz,\, z}(kt),\, F_{ABz,\, z}(t),\, 1,\, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{ABz,\, z}(kt),\, F_{ABz,\, z}(t),\, 1,\, 1) \geq 0.$$

Using (b), we get

$$F_{ABz,\, z}(t) = 1, \text{ for all } t > 0,$$

i.e. $ABz = z$.

Step 3. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz,\, Mx_{2n+1}}(kt),\, F_{ABz,\, STx_{2n+1}}(t),\, F_{Lz,\, ABz}(t),\, F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$\phi(F_{Lz,\ z}(kt), F_{ABz,\ z}(t), F_{Lz,\ ABz}(t), F_{z,\ z}(kt)) \geq 0$

$\phi(F_{Lz,\ z}(kt), 1, F_{Lz,\ z}(t), 1) \geq 0.$

As ϕ is non-decreasing in the first argument, we have

$\phi(F_{Lz,\ z}(kt), 1, F_{Lz,\ z}(t), 1) \geq 0.$

Using (a), we get

$F_{z,\ Lz}(kt) = 1$, for all $t > 0$,

i.e. $z = Lz$.

Thus, we have $z = Lz = ABz$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$\phi(F_{LBz,\ Mx_{2n+1}}(kt), F_{ABBz,\ STx_{2n+1}}(t), F_{LBz,\ ABBz}(t), F_{Mx_{2n+1},\ STx_{2n}}(kt)) \geq 0.$

Letting $n \to \infty$, we get

$\phi(F_{Bz,\ z}(kt), F_{Bz,\ z}(t), F_{Bz,\ Bz}(t), F_{z,\ z}(kt)) \geq 0$

$\phi(F_{Bz,\ z}(kt), F_{Bz,\ z}(t), 1, 1) \geq 0.$

As ϕ is non-decreasing in the first argument, we have

$\phi(F_{Bz,\ z}(t), F_{Bz,\ z}(t), 1, 1) \geq 0.$

Using (b), we have

$F_{Bz,\ z}(t) = 1$, for all $t > 0$,

i.e. $z = Bz$.

Since $z = ABz$, we also have

$z = Az$.

Therefore, $z = Az = Bz = Lz$.

Step 5. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that

$z = Lz = STv.$

Putting $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$\phi(F_{Lx_{2n},\ Mv}(kt), F_{ABx_{2n},\ STv}(t), F_{Lx_{2n},\ ABx_{2n}}(t), F_{Mv,\ STv}(kt)) \geq 0.$

Letting $n \to \infty$, we get

$\phi(F_{z,\ Mv}(kt), F_{z,\ STv}(t), F_{z,\ z}(t), F_{Mv,\ z}(kt)) \geq 0$

$\phi(F_{z,\ Mv}(kt), 1, 1, F_{z,\ Mv}(kt)) \geq 0$

Using (a), we have

$F_{z,\ Mv}(kt) \geq 1$, for all $t > 0$.

Hence, $F_{z,\ Mv}(t) = 1$.

Thus, $z = Mv$.

Therefore, $z = Mv = STv$.

As (M, ST) is occasionally weakly compatible, we have

$STMv = MSTv.$ Thus, $STz = Mz$.

Step 6. Putting $x = x_{2n}$ and $y = z$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Mz}(t), F_{z, Mz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Mz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Mz}(t) = 1$, we have

$$z = Mz = STz.$$

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) and using Step 5, we get

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Lz, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(t), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Tz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Tz}(t) = 1$, we have

$$z = Tz.$$

Since $Tz = STz$, we also have $z = Sz$.

Hence

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Case II. When L is Continuous

As L is continuous, $L^2x_{2n} \to Lz$ and $L(AB)x_{2n} \to Lz$.

As (L, AB) is compatible map of type (β), so

$$LLx_{2n} \to (AB)(AB)x_{2n} \text{ and } LABx_{2n} \to ABz$$

By uniqueness of limit in Menger space, we have

$$Lz = ABz.$$

Step 8. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz, Mx_{2n+1}}(kt), F_{ABz, STx_{2n+1}}(t), F_{Lz, ABz}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Lz, z}(kt), F_{Lz, z}(t), F_{Lz, Lz}(t), F_{z, z}(kt)) \geq 0$$

$\phi(F_{Lz,\, z}(kt), F_{Lz,\, z}(t), 1, 1) \geq 0.$

As ϕ is non-decreasing in the first argument, we have

$\phi(F_{Lz,\, z}(t), F_{Lz,\, z}(t), 1, 1) \geq 0.$

Using (b), we have

$F_{z,\, Lz}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\, Lz}(t) = 1$

$\Rightarrow \quad z = Lz.$

Therefore,

$z = Lz = ABz.$

Step 9. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$\phi(F_{LBz,\, Mx_{2n+1}}(kt), F_{ABBz,\, STx_{2n+1}}(t), F_{LBz,\, ABBz}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$

Letting $n \to \infty$, we get

$\phi(F_{Bz,\, z}(kt), F_{Bz,\, z}(t), F_{Bz,\, Bz}(t), F_{z,\, z}(kt)) \geq 0$

$\phi(F_{Bz,\, z}(kt), F_{Bz,\, z}(t), 1, 1) \geq 0.$

As ϕ is non-decreasing in the first argument, we have

$\phi(F_{Bz,\, z}(t), F_{Bz,\, z}(t), 1, 1) \geq 0.$

Using (b), we have

$F_{Bz,\, z}(t) \geq 1$, for all $t > 0$.

Thus, $F_{Bz,\, z}(t) = 1$

$\Rightarrow \quad z = Bz.$

Since $z = ABz$, we also have $z = Az$.

Therefore, $z = Az = Bz = Lz.$

Step 10. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that

$z = Lz = STv.$

Putting $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$\phi(F_{Lx_{2n},\, Mv}(kt), F_{ABx_{2n},\, STv}(t), F_{Lx_{2n},\, ABx_{2n}}(t), F_{Mv,\, STv}(kt)) \geq 0.$

Letting $n \to \infty$, we get

$\phi(F_{z,\, Mv}(kt), F_{z,\, STv}(t), F_{z,\, z}(t), F_{Mv,\, z}(kt)) \geq 0$

$\phi(F_{z,\, Mv}(kt), 1, 1, F_{z,\, Mv}(kt)) \geq 0$

Using (a), we have

$F_{z,\, Mv}(kt) \geq 1$, for all $t > 0$.

Hence, $F_{z,\, Mv}(t) = 1.$

Thus, $z = Mv$.

Therefore, $z = Mv = STv$.

As (M, ST) is occasionally weakly compatible, we have

$STMv = MSTv.$

Thus, $STz = Mz$.

Step 11. Putting $x = x_{2n}$ and $y = z$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Mz}(t), F_{z, Mz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Mz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Mz}(t) = 1$, we have

$$z = Mz = STz.$$

Step 12. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) and using Step 5, we get

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{Lz, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Tz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Tz}(t) = 1$, we have

$$z = Tz.$$

Since $Tz = STz$, we also have $z = Sz$.

Hence $Az = Bz = Lz = Mz = Tz = Sz = z$.

Hence, the six self maps have a common fixed point in this case also.

Uniqueness. Let w be another common fixed point of A, B, L, M, S and T; then $w = Aw = Bw = Lw = Mw = Sw = Tw$.

Putting $x = z$ and $y = w$ in (3.1.5), we get

$$\phi(F_{Lz, Mw}(kt), F_{ABz, STw}(t), F_{Lz, ABz}(t), F_{Mw, STw}(kt)) \geq 0$$

$$\phi(F_{z, w}(kt), F_{z, w}(t), F_{z, z}(t), F_{w, w}(kt)) \geq 0$$

$$\phi(F_{z, w}(kt), F_{z, w}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, w}(t), F_{z, w}(t), 1, 1) \geq 0.$$

Using (b), we have

$F_{z,\, w}(t) \geq 1$, for all $t > 0$.

Thus, $F_{z,\, w}(t) = 1$,

i.e., $z = w$.

Therefore, z is a unique common fixed point of A, B, L, M, S & T.

This completes the proof.

Remark 3.1. The above theorem is a generalization of the result of Pant et. al. [9] in the sense that the conditions of semi-compatibility and weak compatibility have been replaced by compatibility of type (β) and occasionally weakly compatible.

REFERENCES

[1]. Cho, Y.J., Murthy, P.P. and Stojakovik, M., Compatible mappings of type (A) and common fixed point in Menger space, Comm. Korean Math. Soc. 7 (2), (1992), 325-339.

[2]. Jain, A. and Chaudhary, B., On common fixed point theorems for semi-compatible and occasionally weakly compatible mappings in Menger space, International Journal of Research and Reviews in Applied Sciences, Vol. 14 (3), (2013), 662-670.

[3]. Jain, A. and Singh, B., Common fixed point theorem in Menger space through compatible maps of type (A), Chh. J. Sci. Tech. 2 (2005), 1-12.

[4]. Jain, A. and Singh, B., A fixed point theorem in Menger space through compatible maps of type (A), V.J.M.S. 5(2), (2005), 555-568.

[5]. Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9(4), (1986), 771-779.

[6] Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.

[7] Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.

[8] Mishra, S.N., Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.

[9] Pant, B.D. and Chauhan, S., Common fixed point theorems for semi-compatible mappings using implicit relation, Int. Journal of Math. Analysis, 3 (28), (2009), 1389-1398.

[10] Patel, R.N. and Patel, D., Fixed points of compatible mappings of type (A) on Menger space, V.J.M.S. 4(1), (2004), 185-189.

[11] Schweizer, B. and Sklar, A., Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.

[12] Sehgal, V.M. and Bharucha-Reid, A.T., Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory 6(1972), 97- 102.

[13] Sessa, S., On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd 32(46), (1982), 146-153.

[14] Singh, M., Sharma, R.K. and Jain, A., Compatible mappings of type (A) and common fixed points in Menger space, Vikram Math. J. 20 (2000), 68-78.

[15] Singh, S.L. and Pant, B.D., Common fixed point theorems in probabilistic metric spaces and extension to uniform spaces, Honam. Math. J. 6 (1984), 1-12.