THE UPPER DOMINATING ENERGY OF A GRAPH

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ABSTRACT

For a graph G, a subset D of a vertices set V (G) is called a dominating set of G if every vertex of V − D is adjacent to some vertices of D. The upper domination number Γ(G) of G is the maximum cardinality of a minimal dominating set in G. In this paper, we are introduced upper dominating energy $E_{UD}(G)$ *of a graph G. We are computed upper dominating energies of some standard graphs and a number of well-known families of graphs. Upper and lower bounds for* $E_{UD}(G)$ *are established.*

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I. INTRODUCTION

In this paper, we are consider a simple graph $G(V, E)$, that nonempty, finite, have no loops no multiple and directed edges. Let G be a graph and let n and m be the number of its vertices and edges, respectively. We refer the reader to [8] for more graph theoretical analogist not defined here. A subset D of vertices set V of G is called a dominating set of G if every vertex $v \in (V - D)$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set in G. Any minimal dominating set in G with maximum cardinality is called upper dominating set. For more details in domination theory of graphs we refer to [9].

The concept of energy of a graph was introduced by I. Gutman [6] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The Eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ of A, assumed in non increasing order, are the Eigen values of the graph G. As A is real symmetric, the Eigen values of G are real with sum equal to zero. The energy $E(G)$ of G is defi d to be the sum of the absolute values of the eigen values of G, i.e.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy see [2, 7, 15]. The basic properties including various upper and lower bounds for energy of a graph have been established in [14, 16], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [4, 5].

Recently C. Adiga et al [1] defined the minimum covering energy, EC (G) of a graph which depends on its particular minimum cover C. Further, minimum dom-inating energy, Laplacian minimum dominating energy and minimum dominating distance energy of a graph G can be found in [11, 12, 13] and the references cited there in.

Motivated by these papers, we are introduced upper dominating energy $E_{UD}(G)$ of a graph G. We are computed upper dominating energies of some standard graphs and a number of well-known families of graphs. Upper and lower bounds for $E_{UD}(G)$ are established. It is possible that the upper dominating energy that we are consid-ering in this paper may be have some applications in chemistry as well as in other areas.

II. THE UPPER DOMINATION ENERGY OF A GRAPH

Let G be a graph of order n with vertex set V (G) = $\{v_1, v_2, ..., v_n\}$ and edge set E(G). A dominating set D of a graph G is upper dominating set if it is a minimal dominating set with maximum cardinality. The upper domination number Γ(G) is the maximum cardinality of a minimal dominating set in G. The upper dominating matrix of G is the n × n matrix AUD (G) = (a_{ij}), where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in D; \\ 0, & \text{othewise.} \end{cases}$$

The characteristic polynomial of $A_{UD}(G)$ is denoted by

$$f_n(G, \lambda) := \det(\lambda I - A_{UD}(G)) .$$

The upper dominating Eigen values of the graph G are the Eigen values of $A_{UD}(G)$. Since $A_{UD}(G)$ is real and symmetric, its Eigen values are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The upper dominating energy of

G is defi as:

$$E_{UD}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

We first compute the upper dominating energy of a graph in Figure 1

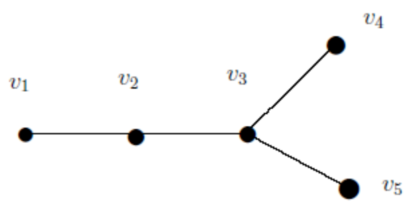

Figure 1

Let G be a graph in Figure 1, with vertices set $\{v_1, v_2, v_3, v_4, v_5\}$ and let its upper dominating set be $D_1 = \{v_1, v_4, v_5\}$. Then

$$A_{UD_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of A_{UD1} (G) is

$$f_n(G, \lambda) = \lambda^5 - 3\lambda^4 - \lambda^3 + 8\lambda^2 - 4\lambda - 1.$$

Hence, the upper dominating Eigen values are $\lambda_1 \approx 1.0000$, $\lambda_2 \approx 2.2631$, $\lambda_3 \approx -0.1827$, $\lambda_4 \approx 1.5157$, $\lambda_5 \approx -1.5962$. Therefore the upper dominating energy of G is

$$E_{UD1}(G) \approx 6.5577$$

If we take another upper dominating set of G, namely D2 = $\{v_2, v_4, v_5\}$, then

$$A_{UD_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of A_{UD2} (G) is

$$fn(G, \lambda) = \lambda^5 - 3\lambda^4 - \lambda^3 + 7\lambda^2 - 2\lambda - 2$$

The upper dominating Eigen values are $\lambda_1 \approx 2.4142$, $\lambda_2 \approx -1.4142$, $\lambda_3 \approx 1.4142$, $\lambda_4 \approx -0.4142$, $\lambda_5 \approx 1.0000$.

Therefore the upper dominating energy of G is

$$E_{UD2}(G) \approx 6.7568.$$

This example illustrates the fact that the upper dominating energy of a graph G depends on the choice of the upper dominating set. i.e. the upper dominating energy is not a graph invariant.

In the following section, we introduce some properties of characteristic polynomials of upper dominating matrix of a graph G.

Theorem 2.1

Let G be a graph of order and size n and m, respectively. Let

$$f_n(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_n$$

be the characteristic polynomials of upper dominating matrix of a graph G. Then

1. $c_0 = 1$,
2. $c_1 = -|D|$
3. $c_2 = \binom{|D|}{2} - m$

Proof

(i) From the definion of $f_n(G, \lambda)$.

(ii) Since the sum of diagonal elements of A_{UD} (G) is equal to |D|, where D is a upper dominating set of a graph G. The sum of determinants of all 1×1 principal submatrices of A_{UD} (G) is the trace of A_{UD} (G) , which evidently is equal to |D|. Thus, $(-1)^1 c_1 = |D|$.

(iii) $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of A_{UD} (G) , that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ij}^2 \\ &= \binom{|D|}{2} - m. \end{aligned}$$

Theorem 2.2

Let G be a graph of order n. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigen values of AUD (G). Then

(i) $\sum_{i}^{n} \lambda_i = |D|$.

(ii) $\sum_{i}^{n} \lambda_i^2 = |D| + 2m$.

Proof

(i) Since the sum of the eigen values of A_{UD} (G) is the trace of A_{UD} (G), then

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$$

(ii) Similarly the sum of squares eigen values of A_{UD} (G) is the trace of $(A_{UD} (G))^2$.

Then

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i\neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= |D| + 2m.\end{aligned}$$

Bapat and S. Pati [3], proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum dominating energy is given in the following theorem.

Theorem 2.3

Let G be a graph with a upper dominating set D. If the upper dominating energy $E_{UD}(G)$ of G is a rational number, then

$$E_D(G) \equiv |D| \pmod 2.$$

Proof

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be upper dominating Eigen values of a graph G of which $\lambda_1, \lambda_2, ..., \lambda_r$ are positive and the rest are non-positive, then

$$\begin{aligned}\sum_{i=1}^{n} |\lambda_i| &= (\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + ... + \lambda_n). \\ &= 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_1 + \lambda_2 + ... + \lambda_n). \\ &= 2q - |D|. \ Where \ q = \lambda_1 + \lambda_2 + ... + \lambda_r.\end{aligned}$$

Therefore, $E_{UD}(G) = 2q - |D|$, and the proof is completed.

III. THE UPPER DOMINATING ENERGY OF SOME GRAPHS

In this section, we investigate the exact values of the upper dominating energy of some standard and well known graphs.

Theorem 3.1

For $n \geq 2$, the upper dominating energy of complete graph Kn, is

$$E_{UD}(K_n) = (n-2) + \sqrt{n^2 - 2n - 5}.$$

Proof

For complete graphs K_n the upper domination number is equal to the domination number (namely one.). Hence, for complete graphs the upper dominating matrix is same as minimum dominating matrix [13]. Therefore the upper dominating energy E_{UD} (K_n) is equal to the minimum dominating energy E_D (K_n).

Theorem 3.2

For the complete bipartite graph $K_{r,s}$, $2 \leq r \leq s$, the upper dominating energy is

$$E_{UD}(K_{r,s}) = (s-1) + \sqrt{1+4rs}.$$

Proof

For the complete bipartite graph $K_{r,s}$, $(2 \le r \le s)$ with vertex set $V = \{v_1, v_2, \bullet\bullet\bullet, v_s, u_1, u_2, \bullet\bullet\bullet, u_r\}$. The upper dominating set is $D = \{v_1, v_2, \bullet\bullet\bullet, v_s\}$.

Then

$$A_{UD}(K_{r,s}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_{UD}(K_{r,s})$, where $n = r + s$ is

$$f_n(K_{r,s}, \lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \lambda^{r-1}(\lambda - 1)^{s-1} (\lambda^2 - \lambda - rs) .$$

The spectrum of $K_{r,s}$ will be written as

$$UD\ Spec(K_{r,s}) = \begin{pmatrix} 0 & 1 & \frac{1+\sqrt{1+4rs}}{2} & \frac{1-\sqrt{1+4rs}}{2} \\ r-1 & s-1 & 1 & 1 \end{pmatrix}$$

Therefore, the upper dominating energy of a complete bipartite graph is

$$E_{UD}(K_{r,s}) = (s-1) + \sqrt{1+4rs}.$$

Theorem 3.3

For $n \ge 2$, the upper dominating energy of a star graph $K_{1,n-1}$ is

$$E_{UD}(K_{1,n-1}) = (n-2) + \sqrt{4n-3}.$$

Proof

Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \bullet\bullet\bullet, v_{n-1}\}$, where v0 is the centre vertex. Then the upper dominating set of $K_{1,n-1}$ is $D = \{v_1, v_2, ..., v_{n-1}\}$.

Hence, the upper dominating matrix of $K_{1,n-1}$ is

$$A_{UD}(K_{1,n-1}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_{UD}(K_{1,n-1})$ is

$$f_n(K_{1,n-1},\lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}$$
$$= (\lambda-1)^{n-2}\left[\lambda^2 - \lambda - (n-1)\right].$$

The spectrum of $K_{1,n-1}$ is

$$UD\ Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{1+\sqrt{1+4(n-1)}}{2} & \frac{1-\sqrt{1+4(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore, the upper dominating energy of a star graph is

$$E_{UD}(K_{1,n-1}) = (n-2) + \sqrt{4n-3}.$$

Definition 3.4

The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers vertices v_0 and u_0 by an edge. A vertex set $V(S_{n,m}) = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{m-1}\}$ and edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j : 1 \le i \le n-1, 1 \le j \le m-1\}$. Therefore, double star graph is bipartite graph.

Theorem 3.5

For $m \ge 3$, the upper dominating energy of double star graph $S_{m,m}$ is

$$E_{UD}(S_{m,m}) = (2m-4) + 2\sqrt{m} + 2\sqrt{m-1}.$$

Proof

For the double star graph $S_{m,m}$ with vertex set $V = \{v_0, v_1, ..., v_{m-1}, u_0, u_1, ..., u_{m-1}\}$ the upper dominating set is $D = \{v_1, v_2, ..., v_{m-1}, u_1, u_2, ..., u_{m-1}\}$. Then

$$A_{UD}(S_{m,m}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{2m \times 2m}$$

The characteristic polynomial of $A_{UD}(S_{m,m})$ is

$$f_m(S_{m,m},\lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda-1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \lambda-1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda-1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda & -1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda-1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)^{2m-4}\ (\lambda^2 - m)\ (\lambda^2 - 2\lambda - (m-2))\ .$$

Hence,

$$UD\ Spec(S_{m,m}) = \begin{pmatrix} 1 & \sqrt{m} & -\sqrt{m} & 1+\sqrt{m-1} & 1-\sqrt{m-1} \\ 2m-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the upper dominating energy of double star graph is

$$E_{UD}(S_{m,m}) = (2m-4) + 2\sqrt{m} + 2\sqrt{m-1}.$$

Definition 3.6

The cocktail party graph, denoted by $K_{2\times p}$, is a graph having vertex set $V(K_{2\times p}) = \bigcup_{i=1}^{p}\{u_i, v_i\}$ and edge set $E(K_{2xp})$ = $\{u_iu_j , v_iv_j , u_iv_j , v_iu_j : 1 \le i < j \le p\}$. i.e. n = 2p, $m = \frac{p^2-3p}{2}$ and forever $v \in V(K_{2\times p})$, $d(v) = 2p - 2$.

Theorem 3.7

For the cocktail party graph $K_{2\times p}$ of order n = 2p, p ≥ 3, the upper dominating energy is

$$E_{UD}(K_{2\times n}) = (2p-3) + \sqrt{4p^2 - 4p - 9}.$$

Proof

For cocktail party graphs $K_{2\times n}$ the upper domination number $\Gamma(K_{2\times n})$ is equal to the domination number $\gamma(K_{2\times n})$ (namely two.). Hence, for cocktail party graphs the upper dominating matrix is same as minimum dominating matrix [13]. Therefore the upper dominating energy E_{UD} $(K_{2\times n})$ is equal to the minimum domi- nating energy E_D (K_n).

Definition 3.8

The crown graph S^0_n for an integer n ≥ 3 is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_iv_i : 1 \le i, j \le n, i \ne j\}$. Therefore S^0_n coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.

Theorem 3.9

For n ≥ 3, the upper dominating energy of the crown graph S^0_n is

$$E_{UD}(S_n^0) = (n-1)\sqrt{5} + \sqrt{4n^2 - 8n + 5}.$$

Proof

For the crown graph S^0_n with vertex set $V = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$, the upper dominating set is $D = \{u_1, u_2, ..., u_n\}$. Then the upper dominating matrix of crown graph is

$$A_{UD}(S_n^0) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n}$$

The characteristic polynomial is

$$f_n(S_n^0, \lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= (\lambda^2 - \lambda - 1)^{n-1}(\lambda^2 - \lambda - (n-1)^2)$$

Hence,

$$UD\ Spec(S_n^0) = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{4n^2-8n+5}}{2} & \frac{1+\sqrt{4n^2-8n+5}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

Therefore

$$E_{UD}(G) = (n-1)\sqrt{5} + \sqrt{4n^2 - 8n + 5}.$$

IV. BOUNDS FOR UPPER DOMINATION ENERGY OF A GRAPH

In this section we shall investigate with some bounds for upper dominating energy of a graph.

Theorem 4.1

Let G be a connected graph of order n and size m. Then

$$\sqrt{2m + \Gamma(G)} \leq E_{UD}(G) \leq \sqrt{n(2m + \Gamma(G))}$$

Proof

Consider the Couchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$, we get

$$(E_{UD}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right)$$
$$\leq n(2m + |D|)$$
$$\leq n(2m + \Gamma(G)).$$

Therefore, the upper bound is hold. For the lower bound, since

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \geq \sum_{i=1}^{n} \lambda_i^2.$$

Then

$$(E_{UD}(G))^2 \geq \sum_{i=1}^{n} \lambda_i^2 = 2m + |D| = 2m + \Gamma(G).$$

Therefore

$$E_{UD}(G) \geq \sqrt{2m + \Gamma(G)}.$$

Similar to Mc Clellands [16] bounds for energy of a graph, bounds for $E_{UD}(G)$ are given in the following theorem.

Theorem 4.2

Let G be a connected graph of order and size n and m, respectively. If P = det(A_{UD}(G)), then

$$E_{UD}(G) \geq \sqrt{2m + \Gamma(G) + n(n-1)P^{2/n}}.$$

Proof Since

$$(E_{UD}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)\left(\sum_{i=1}^{n} |\lambda_i|\right) = \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{i\neq j} |\lambda_i||\lambda_j|.$$

Employing the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)} \sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

Thus

$$\begin{aligned}(E_{UD}(G))^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=j} |\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]} \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left|\prod_{i\neq j} \lambda_i\right|^{2/n} \\ &= 2m + \Gamma(G) + n(n-1)P^{2/n}.\end{aligned}$$

This Complete the proof

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