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RECENT ADVANCEMENT IN FRACTIONAL CALCULUS

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ABSTRACT

Mathematical modelling of various real life scenarios in engineering and sciences leads to differential equations. These traditional models based on integer order derivative may introduce large errors. Fractional calculus helps in reducing this error using fractional derivatives, and has capabilities to provide excellent depiction of memory and heredity properties of processes. In this review paper, we presentthe expressive power of fractional calculus by analyzing two examples viz., mortgage problem and fractional oscillator. These examples help in justifying the advantage of fractional calculus over its integer counterpart. We also present the state-of-the-art of fractional calculus by reviewing the rapid growth of its applications in various domains.

Keywords: Differintegral equation, Fractional calculus, Fractional Classical Mechanics, Laplace Transform, Mortgage Problem, Mittag-Leffler function.

I. INTRODUCTION

Fractional Calculus deals with non-integer order derivatives (called fractional derivatives) and integrals (called fractional integrals). It is not a new concept; the area of fractional calculus was introduced in 1695 by a quest of L'Hospitalwhere he raised the question, "Can the meaning of derivative with integral order be generalized to derivative with non-integral orders?" *i.e.*, what should be a meaningful interpretation of the term $d^{\frac{1}{2}}y/dx^{\frac{1}{2}}$. Leibniz apprehended that "it would be an apparent paradox, from which one day useful consequences will be drawn".Subsequently, Euler and Leibnitz mentioned the derivative of arbitrary order in their work, F. S. Lacroix [1] first obtained the formula for derivative of arbitrary order for the function$y = x^m$ as late as in 1819. He, by showing:

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}, \quad (1)$$

obtained the first characterization for the term $d^{\frac{1}{2}}y/dx^{\frac{1}{2}}$ on taking m = 1 and n = ½:

$$\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \quad (2)$$

Further, many famous mathematicians like N. H. Abel, J. Fourier, J. Liouville and others [2] contributed to the development of the area. For many years the area wasonly considered as a pure theoretical field of mathematics. In the last few decades, numerous applications and physical manifestations of fractional calculus have been developed. Fractional calculus has its basis in fields including mechanics, chemistry, biology, economics, control theory, robotics and image processing. Some of the notable works in these areas that can be mentioned are fractional edge detection [3] in image processing, robot trajectory- control and analysis [4] in robotics, fractional order controllers -- its analysis and synthesis [5,6] in control theory,regular variation in thermodynamics [7], Fractional time evolution [8], in Physics, Discrete random walk model for space-time fractional diffusion [9],Fractional kinetics [10] in chemistry, cardiac tissue electrode interface [11] in biology. speech signals [12] in signals.

Though fractional calculus has found an eminent place in science and engineering, there are still some unresolved challenges. For example, physical interpretation of the fractional derivative is mostly defined based on the application domain. In some cases, it becomes difficult to find a meaningful interpretation for fractional derivative. Finding a well defined interpretation for it, in general, is still an open problem.

In this review, we aim to present some state-of-the-art applications of fractional calculus in science and engineering by showing advantages we gain by using fractional calculus over its integer counterpart. In section 2, we discuss necessary mathematical foundation of the fractional calculus useful to discuss the models as in subsequent sections. In section 3, we highlight applications in fractional differential equations. In section 4, we discuss some notable work using fractional integral equations. In section 5, we understand the importance of the fractional difference equation by highlighting some recent advancement in the area. We conclude by showing the importance of fractional calculus over its integer counterpart.

II. MATHEMATICAL FOUNDATION OF FRACTIONAL CALCULUS

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals for the fractional order, called fractional derivative and integrals *alias* differintegrals. The differintegral is defined as an operator that represents both integer order derivative and integral for positive and negative integer power respectively. If the power is a fraction, it represents a fractional differentiation and integral.

We now define the fractional integral as given by Riemann-Liouville (RL). As integration and differentiation are inverse processes, in (3), $D_a^{-\alpha}$ (which is same as $(D_a^{\alpha})^{-1}$) represents integral of fractional order [13,38].

Definition 1 *Riemann-Liouville integral* of order$\alpha > 0$, for a real valued function $f(t)$ is defined as:

$$I^{\alpha} f(t) = D_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau, t > a. \quad (3)$$

We next give the formula for fractional differentiation. Clearly, if $\alpha \in \mathbb{N}$, it is an ordinary differentiation. For fractional value of $\alpha \in (n-1, n)$, the formula has been derived from Cauchy's formula for n-fold integration. The symbol RL, as in the formula below,indicates that the derivative is a Riemann-Liouville fractional derivative.

Definition 2*Riemann-Liouville fractional derivative* of order $\alpha > 0$ for a real valued function $f(t)$is defined as:

$$^{RL}D_a^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \; n-1 < \alpha < n, n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), \alpha = n \in \mathbb{N} \end{cases} \quad (4)$$

RL fractional derivative have certain limitations. Derivative of a constant function is zero. This property is not satisfied by the RL fractional derivative. The other limitation of RL derivative is that it does not go well while solving a fractional differential equation using Laplace Transform. Evaluation offractional orderderivative at initial point has no physical significance. Finding such fractional derivative at zero is required while transforming a differential equation to s-domain using Laplace Transform. These problems have been resolved by M.Caputoby adjusting the differentiation and the integration in RL fractional derivative (4), thus proposing Caputo fractional derivative as given in (5). It is worth noting that (5) resolve the above mentioned problems. The symbolc, as in (5), indicatesthat it is a Caputo fractional derivative.

Definition 3*Caputo fractional derivative* of order $\alpha > 0$ for a real valued function is:

$$ {}^{c}D_{a}^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{n}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \; n-1 < \alpha < n, n \in \mathbb{N} \\ \frac{d^{n}}{dt^{n}} f(t), \alpha = n \in \mathbb{N} \end{cases} \quad (5)$$

Often, the solution of a fractional differential equation or a fractional integral equation comes in form of an infinite series. Mittag-Leffler [14, 15] in 1903, suggested a generalization of the exponential function as a Mittag-Leffler (ML) function using fractional order. ML function plays crucial role in simplification and analysis of a solution of the differintegral equation.

Definition 4: *Mittag-Leffler function* of one parameter α is:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)} Re(\alpha > 0) \; z \in C. \quad (6)$$

A generalization of Mittag-Leffler function with two parameters α and $\beta$is:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+\beta)}. \qquad Re(\alpha > 0), z, \beta \in C \quad (7)$$

For properties and other generalization of ML function [13, 16] can be referred.

The Laplace Transform method has been used for solving the differintegal equations discussed in section 3. Laplace Transform (LT) is one of the useful techniques used to solve the differintegral equations.We next discuss the LT of RL and Caputo derivative which is useful in solving the differintegral equations [18].

Definition 5:*Laplace transforms of Riemann-Liouville integral and derivative* are given by:

$$\mathcal{L}\{D_{0}^{-\alpha} f(t)\} = s^{-\alpha} F(s)$$

$$\mathcal{L}\{D_{0}^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{k} D_{0}^{\alpha-k-1} f(0)$$

Laplace transform of Caputo derivative is given by

$$\mathcal{L}\{{}^{c}D_{0}^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{k}(0).$$

III. SOME APPLICATIONS OF FRACTIONAL CALCULUS

In the following two subsections, we understand theapplication of fractional calculus *via* fractional differintegral equation in various domains. In section 3.1, mortgage problem has been discussed through a differential equation. Using the mathematical framework, as in section 2, we write the equivalent fractional differential equation withRL-derivative and its solution using LT. We compare the solution with its integer counterpart and understand the effect of variation of fractional parameter α over the monthly instalment, the debt amount and the debt interval. In section 3.2,we discuss the motion of the fractional oscillator through an integral equation using Caputo derivative. We discuss the effect of α on solution and show that fractional order model is more expressible than its integer counterpart.

3.1 Mortgage Problem

Consider a case where a mortgage for amount Rs p is given at a fixed interest rate of r% per month which needs to be paid back, with interest, in n months. The objective is to find the monthly instalment, say Rs X, so that the total due is cleared within specified time. If the total debt at the end of t months is denoted by $f(t)$, then $f(t)$ satisfies the following difference equation:

$$f(t+\Delta t) = f(t)(1+r\Delta t) - X.\Delta t$$

where $X.\Delta t$ is the instalment amount paid in Δt fraction of time. On rearranging the terms and on taking the limit $\Delta t \to 0$, we get the following differential equation:

$$Df(t) = rf(t) - X; \quad f(0) = p \qquad (8)$$

where $D = \frac{d}{dt}$. The solution of (8) is:

$$f(t) = pe^{rt} - \frac{X}{r}(e^{rt} - 1) \qquad (9)$$

Since the debt at time $t = n$ is zero, i.e., $f(n) = 0$, we get the monthly instalment amount as:

$$X = \frac{pre^{rn}}{(e^{rn} - 1)} \qquad (10)$$

For small value of r, weconsider $e^{r} \approx (1+r)$; and therefore from (10) we get:

$$X = \frac{pr(1+r)^n}{(1+r)^n - 1}.$$

Consider the fractional version of mortgage problem as given by (11) and its solution as obtained in [17] using LT given by (12):

$$D^{\alpha}f(t) = rf(t) - X \text{ for } 0 < \alpha < 1 \qquad (11)$$

$$f_{\alpha}(t) = pt^{\alpha-1}E_{\alpha,\alpha}(rt^{\alpha}) - Xt^{\alpha}E_{\alpha,\alpha+1}(rt^{\alpha}) \qquad (12)$$

The two solutions as in (9) and (12)are same for $\alpha = 1$. This can be verified easilyusing $E_{1,1}(rt) = e^{rt}$ and $E_{1,2}(rt) = \frac{1}{rt}\{e^{rt} - 1\}$ (refer (7))

As the debt should be zero at time $t = n$ for any $\alpha \in (0,1]$, i.e., $f_{\alpha}(n) = 0$, from (12), we get the monthly instalment amountas:

$$X = \frac{pE_{\alpha,\alpha}(rn^{\alpha})}{nE_{\alpha,\alpha+1}(rn^{\alpha})} \qquad (13)$$

For $\alpha = 1$, the two instalment amounts as in (10) and (13) are same.

To understand the effect of α on the monthly instalment amount in (13) and on the repayment interval, we consider the following case.

Example 1: Mr A has taken a loan of amount Rs 1, 00,000 at the rate of 12% per year for 4 years. In the following table we mention the monthly instalment where the interest is compounded in different time intervals and by varying the value of the parameter α.

Interest compounded	Monthly Instalment					
	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$	$\alpha = 0.6$	$\alpha = 0.5$
Monthly	2623.18	2252.32	1923.78	1628.57	1358.93	1108.45
Quarterly	2617.49	2248.35	1921.08	1626.79	1357.8	1107.77
Semi-annually	2609.23	2242.59	1917.17	1624.22	1356.17	1106.77
Annually	2593.66	2231.72	1909.80	1619.37	1353.08	1104.90

Table 1: Effect of 'α'and 'interest compounding interval' on instalment amount.

From Table 1, if the interest is compounded monthly, then the instalment amount is higher as compared to the amount when it is compounded quarterly, semi-annually or annually.The effect of αon the instalment amount can be compared withthe effect of interest compounding interval (i.e., monthly, quarterly, etc) on the instalment amount. For any row, say row number one where interest is compounded monthly, the instalment amount decreases as the value of α decreases. We get the same effect if we move along the column from top to down *i.e.*,with the increase in compounding interval.Thus, the effect of decreasing α is same as that of increasing the compounding interval (on the instalment amount).

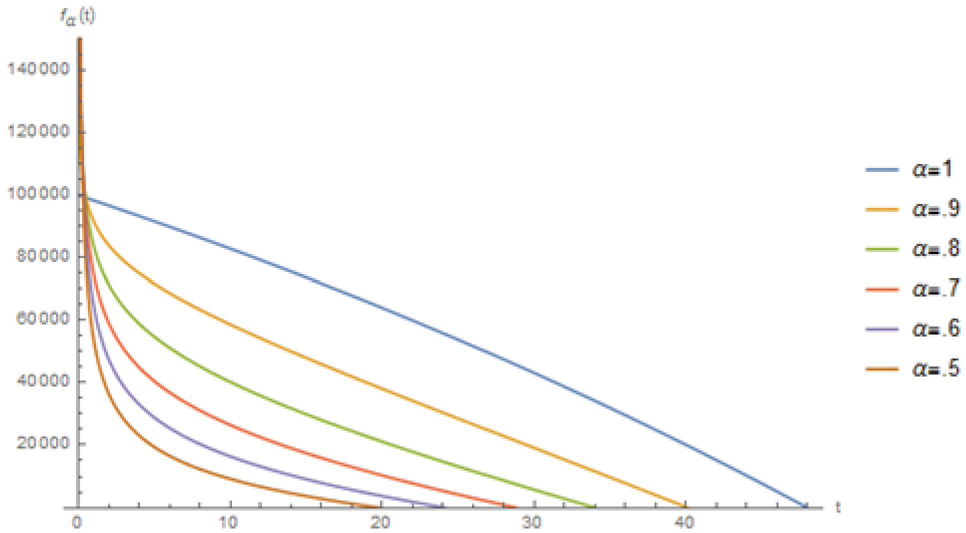

Figure 1: Effect of α on repayment interval

We next discuss the effect of α on repayment interval. For this we consider the (fixed) monthly instalmentRs 2623.18 (monthly instalment when interest is also compounded monthly) and change the value ofα from 0.5 to 1 at the gap of 0.1. The debt amount with respect to time t is shown in the Fig.1.The uppermost curve is for $\alpha = 1$ and the lowermost curve is for$\alpha = 0.5$. From the figure, the debt is repaid much earlier for smaller values of α. This shows that if the fractional model is used, the total repayment amount will be less, and the debt will be paid earlier giving more advantage to the customer. Thus, the smaller value of $\alpha$is equivalent to shortening the interval between payments, and therefore having low effective interest rate.

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3.2 Motion of the fractional oscillator

In classical mechanics, the equation of simple harmonic oscillator plays a very important role as it can be used as an approximation of many other phenomena in quantum physics, electrodynamics, lattice vibrations and phonons. The equation of motion of the simple harmonic oscillator is:

$$m\frac{d^2x}{dt^2} + kx = 0$$

having equivalent integral equation [18]:

$$x(t) = x(0) + x'(0)t - \omega^2 \int_0^1 (t-u)\,x(u)\,du \quad (14)$$

where $x(0)$ and $x'(0)$ are the initial displacement and initial velocity respectively; and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency.

A generalization of the integral appearing in the right hand side of (14) to a fractional integral of order $\alpha > 0$, yield the equation of motion of fractional oscillator as:

$$x(t) = x(0) + x'(0)t - \frac{\omega^2}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} x(u)\,du, \quad 1 < \alpha \leq 2. \quad (15)$$

It is easy to observe that for $\alpha = 2$ both the above equations, i.e. (14) and (15),are same. On solving equation (15) of motion of fractional harmonic oscillator using Laplace transform [18], we get the solution,

$$x(t) = x(0)E_{\alpha,1}(-\omega^2 t^\alpha) + x'(0)tE_{\alpha,2}(-\omega^2 t^\alpha), \quad (16)$$

where $E_{\alpha,1}(-\omega^2 t^\alpha)$ and $E_{\alpha,2}(-\omega^2 t^\alpha)$ are Mittag-Leffler functions (refer (7)).

To understand the variationsin fractional oscillator (FO), i.e., the effect of α on the solution, we consider an example with the initial conditions as:

$$x(0) = x_0, \quad x'(0) = 0 \quad (17)$$

In (16) on applying conditions as in (17), we get displacement of FO as,

$$x(t) = x_0 E_{\alpha,1}(-\omega^2 t^\alpha). \quad (18)$$

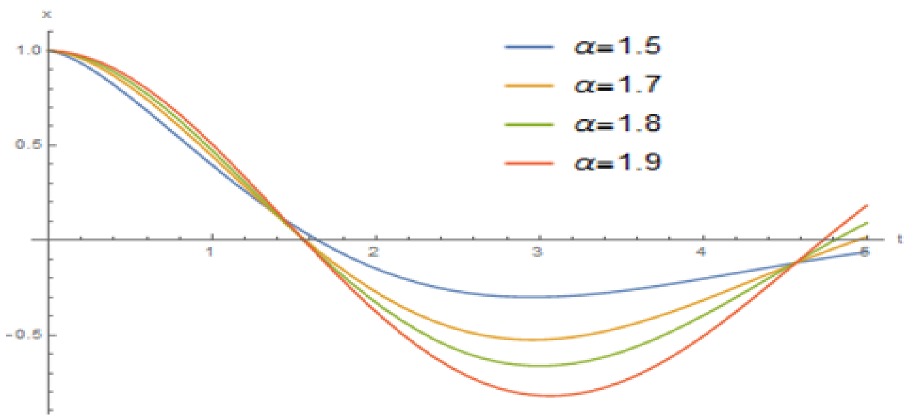

Figure 2: Effect of α on the displacement of FO

Numerical calculations have been done for the displacement of FO by varying α between 1.5 and 1.9; and obtained results are plotted in the Fig.2. It is clear from the figure that the displacement of FO varies on changing α.It behaves like a damped harmonic oscillator for $\alpha < 2$. It is to observe that this effect of damped motion is without using any external forces to the FO. This is unlikely in the case of a simple harmonic oscillator, and therefore, provides more expressive power to FO.

IV. RECENT ADVANCEMENT IN ENGINEERING AND SCIENCE

Many applications of fractional calculus have been developed in areas of science and engineering in the last two decades. Some of the interesting works are being discussed in this section.

4.1 Fractional Differentiation based image processing:

Fractional calculus is recently being shown to be useful in various image processing techniques such as edge detection[19] and texture segmentation, revealing faint objects[20,21], image restorationetc. Author in [22] have shown how to remove the incoherent light scattering produced by a random medium from image, using fractional Fourier Transform. Results have shown substantial improvement in quality of astronomical images. Edge detection often make use of integer order differentiation operators, order 1 used by the gradient and order 2 used by the Laplacian. In [19], authors have shown the advantage of fractional order derivatives to improve the criteria of thin detection and criteria of immunity to noise. Texture Enhancement of medical images is being proposed in [23] using fractional differential mask. Results show that the fractional differential operator can extract subtle information thus improving textual performance.

4.2 Fractional calculus in Biosciences

Fractional calculus models suggest new experiments and measurements that can shed light on the meaning of biological system structure and dynamics. Various recent works have shown the advantages of considering fractional order derivatives in biosciences. In [24], the authors have studied cancer tumor cells based on a fractional diffusion model taking into account spatial and time dependence of concentration of tumor cells as well as that of the killing rate. The authors first studied the approximate analytical solutions using one of the most powerful analytical techniques called q-homotopy analysis method (q-HAM) and from these results, more insight to the description of the need for fractional order observed. They further studied from the numerical analysis on what could be the appropriate order derivative depending on what is targeted. Two Immune effectors interacting with the cancer cells have been studied using fractional-order model in [25]. Authors have found that the fractional order dynamical systems are more suitable to model the tumor-immune system interactions than their integer order counterpart. In [26], author discusses fractional model in studying Arterial viscoelasticity. Fractional order derivatives can be used to conceive a new component called *spring-pot* that interpolates between pure elastic and viscous behaviours. In the paper, authors modified a standard linear solid model replacing a dashpot with a *spring-pot* of order α. The fractional model in human arterial segments was studied and the results showed an accurate relaxation response during 1-hour with least squares errors below 1%. Fractional orders α were 0.2-0.4, justifying the extra parameter. They further discusses that the adapted parameters allowed predicting frequency responses that were similar to reported Complex Elastic Moduli in arteries. Results indicate that fractional models should be considered as real alternatives to model arterial viscoelasticity. Authors in [27], suggested applying fractional calculus to model the behavior of cells and tissues.They believe that it may unravel the inherent complexity of individual molecules and membranes in a way that leads to an improved understanding of the overall biological function and behavior of living systems.

4.3 Fractional calculus in Robotics

In robotics, calculus has a vital role in defining robot motion such as forward and backward movement, rotation, path and trajectory planning, velocity etc; robot control such as force control, multivariable control, vision based control etc and robot vision such as geometry of image formation, camera calibration etc. Fractional calculus has been shown to improve and generalized well established control methods. Fractional order controller was introduced by Oustaloup [28], who developed CRONE controller in pursuing fractal robustness. Recently Podlubny [29] proposed a generalization of the PID controller involving and integrator of order λ and differentiator of order μ, called $PI^{\lambda}D^{\mu}$- controller. It has been shown in [13] that $PI^{\lambda}D^{\mu}$- controller is more capable of adjusting the dynamical properties of control system.

4.4 Fractional Calculus in Analytical Science:

Fractional calculus has recently been shown useful in areas such as milk adulteration [30], ghee adulteration [31] etc. Milk adulteration is an important issue, economically and as well as health point of view. In literature, there have been developed many methods to detect the adulteration of milk. In [30, 32], electrical admittance spectroscopy has been used for detecting fat and water content in milk and freezing point osmometryof milk scheme has been used for measuring the concentration of water in [33]. There are several methods for detection of presence of whey or urea in adulterated milk[34, 35]. The procedure of these methods is neither simple nor quicker and are also expensive. To overcome from these difficulties a constant phase angle based sensor; based on fractional exponent has been developed in [36]. The author has studied the performance of this sensor for detection of milk adulterated with water, liquid whey, urea; and also differentiate between milk and synthetic milk. In [37], fractional order element based impedance sensor also has been introduced which is less expensive and robust.

V. CONCLUSION

It this review paper, we present a basic introduction to fractional calculus, along with two case studies, and its application in various domains. We first introduce some basic definitions and formulae which are used in the paper in the subsequent sections.We have shown how fractional calculus can add more expressive power to the existing integer models, by analyzing mortgage problem and fractional oscillator. To motivate further research, we introduce the state-of-the-art of fractional calculus and its applicability in various engineering and science domains.

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