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CONTINUATION METHODS FOR CONTRACTIVE AND NON EXPANSIVE MAPPING (FUNCTION)

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ABSTRACT

It is concerned with continuation methods for contractive and non expansive maps. We show initially that the property of having a fixed point is invariant by homotopy for contractions. Using this result a nonlinear alternative of Leray–Schauder type is presented for contractive maps and subsequently generalized for nonexpansive maps. An application of the nonlinear alternative for contractions is demonstrated with a second order homogeneous Dirichlet problem. Fixed point theory for continuous, single valued maps in finite and infinite dimensional Banach spaces with a discrete boundary value problem.

II. INTRODUCTION

We begin this paper by showing that the property of having a fixed point is invariant by homotopy for contractions.

Let (X, d) be a complete metric space and U an open subset of X.

Definition 1.1 Let $F : U \to X$ and $G : U \to X$ be two contractions;

here U denotes the closure of U in X.

We say that F and G are homotopic

if there exists $H : U \times [0, 1] \to X$ with the following properties:

(a) $H(\cdot, 0) = G$ and $H(\cdot, 1) = F$;

(b) $x = H(x, t)$ for every $x \in \partial U$ and $t \in [0, 1]$ (here ∂U denotes the boundary of U in X);

(c) there exists α, $0 \le \alpha < 1$, such that $d(H(x, t),H(y, t)) \le \alpha\, d(x, y)$ for every $x, y \in U$ and $t \in [0, 1]$;

(d) there exists M, $M \ge 0$, such that $d(H(x, t),H(x, s)) \le M\, |t - s|$ for every $x \in U$ and $t, s \in [0, 1]$.

Theorem 1.1 Let (X, d) be a complete metric space and U an open subset of X. Suppose that $F : \bar{U} \to X$ and $G : \bar{U} \to X$ are two homotopic

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contractive maps and G has a fixed point in U. Then F has a fixed point in U.

Proof Consider the set

$A = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in \bar{U}\}$

where H is a homotopy between F and G as described in Definition 3.1.

Notice A is nonempty since G has a fixed point, that is, $0 \in A$. We will

show that A is both open and closed in [0, 1] and hence by connectedness

we have that A = [0, 1]. As a result, F has a fixed point in U.

We first show that A is closed in [0, 1]. To see this let

$$\{\lambda_n\}_{n=1}^{\infty} \quad A \text{ with } \lambda_n \to \lambda \quad [0, 1] \text{ as } n\to\infty.$$

We must show that $\lambda \in A$. Since $\lambda_n \in A$ for n = 1, 2, . . .,

there exists

$x_n \in \bar{U}$ with $x_n = H(x_n, \lambda_n)$. Also for $n,m \in \{1, 2, \ldots\}$

we have

$$d(x_n, x_m) = d(H(x_n, \lambda_n),H(x_m, \lambda_m))$$

$$\leq d(H(x_n, \lambda_n),H(x_n, \lambda_m)) + d(H(x_n, \lambda_m),H(x_m, \lambda_m))$$

$$\leq M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m),$$

that is,

$$d(x_n, x_m) \leq \left(\frac{M}{1-\alpha}\right) |\lambda_n - \lambda_m|.$$

Since $\{\lambda_n\}$ is a Cauchy sequence we have that $\{x_n\}$ is also a Cauchy sequence, and since X is complete there exists $x \in U$ with $\lim_{n\to\infty} x_n = x$ In addition, $x = H(x, \lambda)$ since

$$d(x_n,H(x, \lambda)) = d(H(x_n, \lambda_n),H(x, \lambda))$$

$$\leq M |\lambda_n - \lambda| + \alpha d(x_n, x).$$

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Thus $\lambda \in A$ and A is closed in [0, 1].

Next we show that A is open in [0, 1]. Let $\lambda 0 \in A$. Then there exists

$x0 \in U$ with $x0 = H(x0, \lambda 0)$. Fix $\in > 0$ such that

$\in \leq \frac{(1-\alpha)r}{M}$ where $r < dist(x0, \partial U)$,

and where $dist(x0, \partial U) = \inf\{d(x0, x) : x \in \partial U\}$. Fix $\lambda \in (\lambda 0 - \in, \lambda 0 + \in)$.

Then for $x \in \overline{B(x0, r)} = \{x : d(x, x0) \leq r\}$,

$$d(x0, H(x, \lambda)) \leq d(H(x0, \lambda 0), H(x, \lambda 0)) + d(H(x, \lambda 0), H(x, \lambda))$$

$$\leq \alpha d(x0, x) + M|\lambda - \lambda 0|$$

$$\leq \alpha r + (1 - \alpha)r = r.$$

Thus for each fixed $\lambda \in (\lambda 0 - \in, \lambda 0 + \in)$,

$H(\cdot, \lambda) : B(x0, r) \to B(x0, r)$.

We can now apply Theorem 1.1 (an argument based on Theorem 1.3

could also be used) to deduce that $H(\cdot, \lambda)$ has a fixed point in $\bar{U}$. Thus

$\lambda \in A$ for any $\lambda \in (\lambda 0 - \in, \lambda 0 + \in)$,and therefore A is open in [0, 1].we will assume that X is a Banach space. We now present a nonlinear alternative of Leray–Schauder type for contractive maps.

Theorem 3.2 Suppose U is an open subset of a Banach space X, $0 \in U$

and $F : \bar{U} \to X$ a contraction with $F(\bar{U})$ boded. Then either

(A1) F has a fixed point in $\bar{U}$, or

(A2) there exist $\lambda \in (0, 1)$ and $u \in \partial \bar{U}$ with $u = \lambda F(u)$ holds.

Proof:- Assume (A2) does not hold and F has no fixed points on ∂U

(otherwise we are finished). Then

$u \neq \lambda F(u)$ for all $u \in \partial U$ and $\lambda \in [0, 1]$.

Let $H : \bar{U} \times [0, 1] \to X$ be given by

$H(x, t) = tF(x)$,

and let G be the zero map. Notice G has a fixed point in U (that is, $0 = G(0)$) and F and G are two homotopic, contractive mappings. We can now apply Theorem 3.1 to deduce that there exists $x \in U$ with $x = F(x)$, that is, (A1) occurs. It is natural to ask whether we can extend Theorem 3.2 to non expansive maps as Theorem 2.5 suggests.

Theorem 3.3 Let U be a boded, open, convex subset of a iformly convex Banach space X, with $0 \in U$ and $F : \bar{U} \to X$ a non expansive map. Then either

(A1) F has a fixed point in U, or

(A2) there exist $\lambda \in (0, 1)$ and $u \in \partial \bar{U}$ with $u = \lambda F(u)$ is true.

Proof:- Assume (A2) does not hold. Consider for each $n \in \{2, 3, . . .\}$, the

Mapping $F_n := (1 - \frac{1}{n})F : \bar{U} \to X$.

Notice that Fn is a contraction with contraction constant $1 - 1/n$. Applying Theorem 3.2 to F_n, we deduce that either Fn has a fixed point in U, or there exist $\lambda \in (0, 1)$ and $u \in \partial U$ with $u = \lambda F_n(u)$.

Suppose the latter is true, that is, there exist $\lambda \in (0, 1)$ and $u \in \partial U$ with $u = \lambda F_n(u)$.

Then

$u = \lambda(1 - \frac{1}{n}) F(u) = \eta F(u)$ where $0 < \eta = \lambda(1 - \frac{1}{n}) < 1$

– a contradiction since property (A2) does not occur. Consequently for each $n \in \{2, 3, . . .\}$ we have that Fn has a fixed point $\in U$.

A standard result (if E is a reflexive Banach space, then any norm boded sequence in E has a weakly convergent subsequence) implies (since U is closed, boded and convex – hence weakly closed) that there exist a subsequence S of integers and a $u \in U$ with ; u as $n \to \infty$ in S; here $\to$; denotes weak convergence.

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In addition since $= (1 - 1/n)F(u_n)$ we have

$$\|(I - F)(u_n)\| = \frac{1}{n}\|F(u_n)\|$$

$$\leq \frac{1}{n}\|F(u_n) - F(0)\| + \|F(0)\|$$

$$\leq \frac{1}{n}\|(u_n) - F(0)\|.$$

Thus (I − F)(u_n) converges strongly to 0. The demiclosedness of I – F (see Exercise 2.8) implies that u = F(u), and as a result (A1) occurs.

To illustrate how Theorem 3.2 can be applied in practice we turn our attention to the second order homogeneous Dirichlet problem,(3.1)

$$\begin{cases} y'' = f(t, y, y') \text{ for } t \in [a, b], \\ y(a) = y(b) = 0, \end{cases}$$

where $f : [a, b] \times R^2 \to R$ is continuous. Associated with (3.1) we consider the following related family of problems:

(3.2)λ

$$\begin{cases} y^{11} = \lambda f(t, y, y^1) \text{ for } t \in [a, b], \\ y(a) = y(b) = 0, \end{cases}$$

for λ ∈ (0, 1). Define an operator $F : C^1[a, b] \to C^1[a, b]$ by

$$F_y(t) := \int_a^b G(t,s) f\left(s, y(s), y'(s)\right) ds$$

where the Green's function G(t, s) is given by

$$G(t, s) = \begin{cases} -\frac{(t-a)(b-s)}{b-a}, & a \leq t \leq s \leq b, \\ = -\frac{(s-a)(b-s)}{b-a}, & a \leq s \leq t \leq b. \end{cases}$$

By the properties of the Green's function, the fixed points of F are the classical solutions of (3.1). der an appropriate local Lipschitz condition on f, we will use the nonlinear alternative for contractive maps to establish that F restricted to the closure of a suitable open set U ⊆ $C^1[a, b]$ is contractive and has a fixed point (in fact a unique fixed point) in Ū. Hence (3.1) has a unique solution in Ū.

To this end we assume that f satisfies the following local Lipschitz condition:

(3.3) there are a subset $D \subseteq R^2$ and constants K_0 and K_1 such that f restricted to [a, b] × D satisfies

$$|f(t, y, y') - f(t, z, z')| \le K_0|y - z| + K1|y' - z'|.$$

Define a modified maximum norm on $C^1[a, b]$ by

$$y' = K_0|y|_0+K_1|y'|_0 \text{ where } |y|_0 = \sup_{t\in[a,b]} |y(t)| \text{ and } |y'|_0 = \sup_{t\in[a,b]} |y'(t)|.$$

For functions y and z whose values and derivative values lie in the region where f is locally Lipschitz, we have

$$|(F_y - F_z)(t)| = |\int_a^b G(t,s)f\left(s,y(s),y'(s)\right)ds - f(s, z(s), z'(s))]\, ds|$$

$$\le \frac{(b-a)^2}{8} \,||\, y- z'||$$

since

$$\max_{t\in[a,b]} \int_a^b G(t,s)\,ds = \max_{t\in[a,b]} \frac{(b-t)(t-b)}{2} = \frac{(b-a)^2}{8}$$

Thus

$$|F_y - F_z|_0 \le \frac{(b-a)^2}{8} \,||y' - z'||$$

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for functions y and z whose values and derivative values lie in the region where f is locally Lipschitz. Likewise

$|(F_y - F_z)'|_0 \leq \frac{(b-a)}{2} \|y' - z'\|$

for functions y and z whose values and derivative values lie in the region where f is locally Lipschitz, since

$$\max \int_a^b Gt(t,s)\,ds = \max \frac{(b-a)^2 - (t-b)^2}{2(b-a)} = \frac{(b-a)}{2} t \in [a,b]$$

Consequently

$$(3.4)\ \|Fy - Fz\| \leq \left[k_0 \frac{(b-a)^2}{8} + k_1 \frac{(b-a)}{2}\right]$$

for functions y and z whose values and derivative values lie in the region

where f is locally Lipschitz. This inequality and Theorem 3.2 enable us

to establish the following existence and uniqueness principle for (3.1).

Theorem 3.4 Let $f : [a, b] \times R^2 \to R$ be continuous and satisfy (3.3)

in a set D with constants K0 and K1 such that

$$(3.5) \qquad k_0 \frac{(b-a)^2}{8} + k_1 \frac{(b-a)}{2} < 1$$

is true. Suppose there is a boded open set of functions $U \subseteq C^1[a, b]$ with $0 \in U$ such that

(3.6) $u \in \bar{U}$ implies $(u(t), u'(t)) \in D$ for all $t \in [a, b]$

And (3.7) y solves $(3.2)\lambda$ for some $\lambda \in (0, 1)$ implies y € ∂U hold. Then (3.1) has a unique solution in $\bar{U}$.

Proof Evidently $F : \bar{U} \to C1[a, b]$ is contractive by (3.4) and (3.5).

Apply Theorem 3.2 and note that (A2) cannot occur because of (3.7).

Remark 3.1 In many important applications, the function f is independent of y′; that is f = f(t, y). In this case, a straightforward review of the reasoning given above shows that we can regard F as

F : C[a, b] → C[a, b]. This leads to a useful variant of Theorem 3.4 in which D ⊆ R, all reference to y_ and z_ is dropped in (3.3), and U ⊆ C[a, b].

Example 3.1 The boundary value problem

$$(3.8)\qquad \begin{cases} y''(t) = -e^{y(t)},\ t \in [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

has a unique solution with maximum norm at most 1. We note that(3.8) models the steady state temperature in a rod with temperature dependent internal heating. To establish the above claim we apply Theorem 3.4 and Remark 3.1

with $f = f(t, y) = -e^y$. By the mean value theorem we have that

$|y| \leq 1$ and $|z| \leq 1$ imply $|e^y - e^z| \leq e^{\max\{y,z\}}|y - z| \leq e\,|y - z|$.

We take

$$D = [-1, 1] \text{ and } U = \left\{ y \in C[0, 1] : |y|0 = \sup_{t\in[0,1]} |y(t)| < 1 \right\}$$

in Theorem 3.4. Then

$$\frac{ko}{8} = \frac{e}{8} < 1$$

Suppose that y solves

$$(3.9)\lambda\qquad \begin{cases} y''(t) = -\lambda e^{y(t)},\ t \in [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

for some λ ∈ (0, 1). Then

$$y(t) = -\lambda \int_0^1 G(t,s)ey(s)ds$$

and therefore

$|y(t)| \leq \frac{1}{8} e^{|y|0}$ for $t \in [0, 1]$,

$|y|_0 \leq \frac{1}{8} e^{|y|0}$

II. CONCLUSION

Consequently $|y|0 \leq 1$ and this implies that $|y|0 \neq 1$ and therefore y € ∂U. Now Theorem 3.4 implies that (3.8) has a unique solution with norm at most 1.

REFERENCE

[1] R. P. AGARWAL and D. O'REGAN, Difference equations in abstract spaces, J. Austral. Math. Soc. Ser. A, 64 (1998), 277–284.

[2] R. P. AGARWAL and D. O'REGAN, A note on the existence of multiple fixed points for multi valued maps with application, J. Diff. Eqns, 160 (2000), 389–403.

[3] C. D. ALIPRANTIS and K. C. BORDER, Infinite dimensional analysis, Springer-Verlag, 1994.

[4] D. ALSPACH, A fixed point free nonexpansive map, Proc. Amer. Math. Soc., 82 (1981), 423–424.

[5] M. ALTMAN, A fixed point theorem for completely continuous operators in Banach spaces, Bull. Acad. Polon. Sci., 3 (1955), 409–413.

[6] H. AMANN, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Funct. Anal., 14 (1973), 346–384.

[7] H. AMANN, Fixed point problems and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., 18 (1976), 620–709.

[8] J. APPELL and M. P. PERA, Noncompactness principles in nonlinear operator approximation theory, Pacific J. Math., 115 (1984), 13–31.

[9] J. P. AUBIN and A. CELLINA, Differential inclusions, Springer-Verlag, 1984.

[10] J. P. AUBIN and I. EKELAND, Applied nonlinear analysis, John Wiley and Sons, 1984.

[11] D. F. BAILEY, Some theorems on contractive mappings, J. London Math. Soc., 41 (1966), 101–106.

[12] J. B. BAILLON and R. SCHOENBERG, Asymptotic normal structure and fixed points of non expansive mappings, Proc. Amer. Math. Soc., 81 (1981), 257–264.

[13] J. BANAS and K. GOEBEL, Measures of noncompactness in Banach spaces, Marcel Dekker, 1980.