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FIXED POINT THEOREMS IN FUZZY METRIC SPACE VIA ABSORBING MAPPINGS

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ABSTRACT

In this paper, the concept of absorbing maps in fuzzy metric space has been introduced to prove common fixed point theorems. Our results extend, generalize, fuzzyify several fixed point theorems on metric spaces, Menger Probabilistic Metric spaces, Fuzzy metric spaces as well as the result of Singh et. al. [11] and many. We also cited an example in support of our result.

Keywords: *Common fixed points, fuzzy metric space, weak compatible maps and absorbing maps.*

AMS Subject Classification: *Primary 47H10, Secondary 54H25.*

I. INTRODUCTION

The concept of Fuzzy sets was initially investigated by Zadeh [13] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [8] and modified by George and Veeramani [4]. Recently, Grabiec [5] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [10] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [6] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [2, 3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh et. al. [11, 12] proved fixed point theorems in a fuzzy metric space. In this paper, a fixed point theorem for six self maps has been established using the concept of absorbing maps. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

II. PRELIMINARIES

2.1. Definition [9] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for $a, b, c, d \in [0, 1]$. Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

2.2. Definition. [9] The 3-tuple $(X, M, *)$ is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$, (FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [9] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x,y,t) = \frac{t}{t + d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

2.3. Definition. [9] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.
A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

2.4. Definition. [11] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points.

2.5. Definition. Suppose A and B be two self mappings on a Fuzzy metric space (X, M,*), then A is called B-absorbing if there exists a positive $R > 0$ such that $M(Bx, BAx, t) \geq M(Bx, Ax, t/R)$ for all $x \in X$. Similarly, B is called A-absorbing if there exists a positive $R > 0$ such that $M(Ax, ABx, t) \geq M(Ax, Bx, t/R)$ for all $x \in X$.

Now, we give an example which shows that absorbing map need not commute at their coincidence points.

Example 2.2. Let X = [0, 2] be a metric space and d and M are same as in Example 2.1. Define $A, B : X \to X$ by

$$Ax = \begin{cases} 2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases} \quad \text{and} \quad Bx = 2 \quad \text{for } x \in X.$$

Then the map A is B-absorbing for any $R > 2$ but the pair of maps (A, B) are not commute at their coincidence point $x = 0$.

2.6. Definition. Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be any kind of coincidentally commuting mappings if and only if there is a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$ and $fgu = gfu$ at this point.

Example 2.3. Let (X, M, *) be a Fuzzy metric space, where X = [0, 2] with a t - norm defined by $a * b = \min\{a, b\}$ for all $a, b \in X$ and

$$M(x,y,t) = \begin{cases} \frac{t}{t + |x-y|} & , \text{if } t > 0 \\ 0 & , \text{if } t = 0 \end{cases} \quad \text{for all } x, y \in X.$$

Define $f, g : [0, 2] \to [0, 2]$ by

$$f(x) = \begin{cases} 2, & \text{if } x \in [0,1] \\ \frac{x}{2}, & \text{if } x \in (1,2] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & \text{if } x \in [0,1] \\ \frac{x+3}{5}, & \text{if } x \in (1,2] \end{cases}.$$

Consider the sequence $\{x_n\} = \left(2 - \frac{1}{2n}\right)$. Clearly f(1) = g(1) = 2 and f(2) = g(2) = 1. Also fg(1) = gf(1) = 1 and fg(2) = gf(2) = 2.

Thus, f and g are weakly compatible mappings.

Now $fx_n = \left(1 - \frac{1}{4n}\right)$ and $gx_n = \left(1 - \frac{1}{10n}\right)$.

Therefore, $fx_n \to 1$, $gx_n \to 1$, $fg(x_n) = 2$, $gf(x_n) = \left(\frac{4}{5} - \frac{1}{20n}\right)$ and $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = \frac{t}{t + \frac{6}{5}} \neq 1$, so f and g are not compatible maps on X but they are any kind of coincidentally commuting mappings.

2.1. Remark. The above example shows that weakly compatible mappings are also any kind of coincidentally commuting mappings.

2.1. Lemma. [5] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, M(x, y, .) is a non-decreasing function.

2.2. Lemma. [1] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$

$$M(x, y, kt) \geq M(x, y, t) \ \forall \ t > \quad \text{then} \ x = y.$$

2.3. Lemma. [12] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \ \forall \ t > 0 \text{ and } n \in N.$$ Then $\{x_n\}$ is a Cauchy sequence in X.

2.4. Lemma. [7] The only t-norm * satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is

$$a * b = \min\{a, b\} \text{ for all } a, b \in [0, 1].$$

III. MAIN RESULT

3.1. Theorem. Let (X, M, *) be a complete Fuzzy metric space with continuous t-norm defined by a * b = min{a, b} where $a, b \in [0, 1]$ and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied :

1. $P(X) \subset ST(X)$, $Q(X) \subset AB(X)$;
2. there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$
3. $M(Px, Qy, qt) \geq \min\{M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, t), M(Px, STy, t)\}$;
4. for all $x, y \in X$, $\lim_{t\to\infty} M(x, y, t) = 1$;
5. AB = BA, ST = TS, PB = BP, QT = TQ;
6. Q is ST-absorbing.

If the pair of maps (P, AB) is reciprocal continuous and semi-compatible, then P, Q, S, T, A and B have a unique common fixed point in X.

Proof : Let $x_0 \in X$. From (a), there exist $x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \quad \text{and} \quad Qx_1 = ABx_2 .$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Px_{2n-2} = STx_{2n-1} = y_{2n-1} \quad \text{and}$$

$$Qx_{2n+1} = ABx_{2n} = y_{2n} \quad \text{for } n = 1, 2, 3, \ldots .$$

By using contractive condition (b), we obtain

$$M(Px_{2n}, Qx_{2n+1}, qt) \geq \min\{M(ABx_{2n}, STx_{2n+1}, t), M(Px_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(Px_{2n}, STx_{2n+1}, t)\}$$
$$= \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+1}, t)\}$$
$$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}.$$

From Lemma 2.4, we have

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we have

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Thus, we have

$$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t) \text{ for } n = 1, 2, \ldots$$
$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n+1}, t/q)$$
$$\geq M(y_{n-2}, y_{n-1}, t/q^2)$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\geq M(y_1, y_2, t/q^n) \to 1 \text{ as } n \to \infty, \text{ and hence } M(y_n, y_{n+1}, t) \to 1 \text{ as } n \to \infty \text{ for any } t > 0.$$

For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that

$$M(y_n, y_{n+1}, t) > 1 - \varepsilon \quad \text{for all } n > n_0.$$

For $m, n \in N$, we suppose $m \geq n$. Then we have

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m\text{-}n) * M(y_{n+1}, y_{n+2}, t/m\text{-}n) * \ldots * M(y_{m-1}, y_m, t/m\text{-}n)$$
$$\geq (1 - \varepsilon) * (1 - \varepsilon) * \ldots * (1 - \varepsilon) \text{ (m - n) times}$$
$$\geq (1 - \varepsilon) \text{and hence } \{y_n\} \text{ is a Cauchy sequence in X.}$$

Since (X, M, *) is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converge to the same point i.e. $z \in X$

i.e., $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$ (1)

$\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$. (2)

Since the pair (P, AB) is reciprocally continuous mapping, then we have

$$\lim_{n\to\infty} PABx_{2n} = Pz \quad \text{and} \quad \lim_{n\to\infty} ABPx_{2n} = ABz.$$

And semi-compatibility of (P, AB) gives

$ABPx_{2n} \to ABz$ therefore $Pz = ABz$. (3)

We claim $Pz = ABz = z$.

Step 1. Put $x = z$ and $y = x_{2n+1}$ in (b), we have

$$M(Pz, Qx_{2n+1}, qt) \geq \min\{M(ABz, STx_{2n+1}, t), M(Pz, ABz, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(Pz, STx_{2n+1}, t)\}.$$

Taking $n \to \infty$ and using equation (1), we get

$$M(Pz, z, qt) \geq \min\{M(z, z, t), M(Pz, z, t), M(z, z, t), M(Pz, z, t)\}$$

i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.

Therefore, by using Lemma 2.2, we get

$$Pz = z.$$

Therefore, $ABz = Pz = z$.

Step 2. Putting $x = Bz$ and $y = x_{2n+1}$ in condition (b), we get

$$M(PBz, Qx_{2n+1}, qt) \geq \min\{M(ABBz, STx_{2n+1}, t), M(PBz, ABBz, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(PBz, STx_{2n+1}, t)\}.$$

As $BP = PB$, $AB = BA$, so we have

$$P(Bz) = B(Pz) = Bz \text{ and}$$

$$(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz.$$

Taking $n \to \infty$ and using (1), we get

$$M(Bz, z, qt) \geq \min\{M(Bz, z, t), M(Bz, Bz, t), M(z, z, t), M(Bz, z, t)\}$$

i.e. $M(Bz, z, qt) \geq M(Bz, z, t)$.

Therefore, by using Lemma 2.2, we get

$$Bz = z$$

and also we have

$$ABz = z$$

$\Rightarrow$ $Az = z$.

Therefore, $Az = Bz = Pz = (4)$

Step 3. As $P(X) \subset ST(X)$, there exists $u \in X$ such that

$$z = Pz = STu.$$

Putting $x = x_{2n}$ and $y = u$ in (b), we get

$$M(Px_{2n}, Qu, qt) \geq \min\{M(ABx_{2n}, STu, t), M(Px_{2n}, ABx_{2n}, t), M(Qu, STu, t), M(Px_{2n}, STu, t)\}.$$

Taking $n \to \infty$ and using (1) and (2), we get

$$M(z, Qu, qt) \geq \min\{M(z, z, t), M(z, z, t), M(Qu, z, t), M(z, z, t)\}$$

i.e. $M(z, Qu, qt) \geq M(z, Qu, t)$

Therefore, by using Lemma 2.2, we get

$$Qu = z.$$

Hence $STu = z = Qu$.

Since Q is ST-absorbing then

$$M(STu, STQu, t) \geq M(STu, Qu, t/r) = 1$$

i.e. $STu = STQu$

$\Rightarrow \quad z = STz.$

Step 4. Putting $x = x_{2n}$ and $y = z$ in (b), we get

$$M(Px_{2n}, Qz, qt) \geq \min\{M(ABx_{2n}, STz, t), M(Px_{2n}, ABx_{2n}, t), M(Qz, STz, t), M(Px_{2n}, STz, t)\}.$$

Taking $n \to \infty$ and using (2) and step 3, we get

$$M(z, Qz, qt) \geq \min\{M(z, Qz, t), M(z, z, t), M(Qz, Qz, t), M(z, Qz, t)\}$$

i.e. $M(z, Qz, qt) \geq M(z, Qz, t).$

Therefore, by using Lemma 2.2, we get

$$Qz = z.$$

So, $z = Qz = STz.$

Step 5. Putting $x = x_{2n}$ and $y = Tz$ in (b), we get

$$M(Px_{2n}, QTz, qt) \geq \min\{M(ABx_{2n}, STTz, t), M(Px_{2n}, ABx_{2n}, t), M(QTz, STTz, t), M(Px_{2n}, STTz, t)\}.$$

As $QT = TQ$ and $ST = TS$, we have

$$QTz = TQz = Tz \text{ and}$$

$$ST(Tz) = T(STz) = TQz = Tz.$$

Taking $n \to \infty$, we get

$$M(z, Tz, qt) \geq \min\{M(z, Tz, t), M(z, z, t), M(Tz, Tz, t), M(z, Tz, t)\}$$

$$\geq \min\{M(z, Tz, t), M(z, Tz, t)\}$$

i.e. $M(z, Tz, qt) \geq M(z, Tz, t).$

Therefore, by using Lemma 2.2, we get

$$Tz = z.$$

Now $STz = Tz = z$ implies $Sz = z$.

Hence, $Sz = Tz = Qz =$ (5)

Combining (4) and (5), we get

$$Az = Bz = Pz = Qz = Tz = Sz = z.$$

Hence, z is the common fixed point of A, B, S, T, P and Q.

Uniqueness : Let u be another common fixed point of A, B, S, T, P and Q.

Then $Au = Bu = Pu = Qu = Su = Tu = u.$

Put $x = z$ and $y = u$ in (b), we get

$$M(Pz, Qu, qt) \geq \min\{M(ABz, STu, t), M(Pz, ABz, t), M(Qu, STu, t), M(Pz, STu, t)\}.$$

Taking $n \to \infty$, we get

$$M(z, u, qt) \geq \min\{M(z, u, t), M(z, z, t), M(u, u, t), M(z, u, t)\}$$

$$\geq \min\{M(z, u, t), M(z, u, t)\}$$

i.e. $M(z, u, qt) \geq M(z, u, t).$

Therefore by using Lemma 2.2, we get

$$z = u.$$

Therefore z is the unique common fixed point of self maps A, B, S, T, P and Q.

3.2. Theorem. Let (X, M, *) be a complete Fuzzy metric space with continuous t-norm defined by $a * b = \min\{a, b\}$ where $a, b \in [0, 1]$ and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied :

1. $P(X) \subset ST(X)$, $Q(X) \subset AB(X)$;
2. there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$
3. $M(Px, Qy, qt) \geq \min\{M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, t), M(Px, STy, t)\}$;
4. for all $x, y \in X$, $\lim_{t\to\infty} M(x, y, t) = 1$;
5. AB = BA, ST = TS, PB = BP, QT = TQ;
6. Q is ST-absorbing;

If the pair of maps (P, AB) is subsequential continuous and semi-compatible then P, Q, S, T, A and B have a unique common fixed point in X.

Proof. Since reciprocal continuity implies subsequential continuity, so the proof follows from Theorem 3.1.

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