Finding a Measure, A Ring Theoretical Approach

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ABSTRACT

Let S be a nonempty set and F consists of all Z2 characteristic functions defined on S. We are supposed to introduce a ring isomorphic to (P(S),4,∩), whose set is F. Then, assuming a finitely additive function m defined on P(S), we change P(S) to a pseudometric space (P(S),dm) in which its pseudometric is defined by m. Among other things, we investigate the concepts of convergence and continuity in the induced pseudometric space. Moreover, a theorem on the measure of some kinds of elements in (P(S),m) will be established. At the end, as an application in probability theory, the probability of some events in the space of permutations with uniform probability will be determined. Some illustrative examples are included to show the usefulness and applicability of results

Keywords: Ring, coefficients, topology, probability, set

I. INTRODUCTION

Some of the propositions in mathematics can be used in science or other branches of mathematics to obtain some new results. For instance, a force field isal way savector field representing forces. The laws of conservation of energy, conservation of angular momentum, Kepler's laws of planetary motion in astronomy are some of the consequence softhe mathematical theory of central vector fields [10]. As another example, let us to consider the fundamental theorem of algebra. It states that every non constant single variable polynomial with complex coefficients has at least one complex root. In [2] the authors proved that the Gauss-Bonnet theorem in differential geometry implies the fundamental theorem of algebra. There are also other proofs which involve some analysis, or at least the topological concept of continuity of real or complex functions, differentiable or analytic functions [4], Cauchy's integral theorem [12], or even the theory of compact differentiable manifolds [3, 11] and intersection theory mod 2 [9]. Recently, some another proofs about the existence of the roots for polynomials, also has been given in [5, 6]. In this paper, as a follow up to the method of mentioned works, we consider the Boolean ring (P(S),4,∩) [8] with a measure on P(S) and investigate its properties to find a mathematical formula for the measure of some special kinds of subsets of S. At the end, we apply the previous work to the theory of uniform probability spaces. We are able to include some basic results whose detailed proofs are not easily available elsewhere.

II. PRELIMINARY NOTES

Let S be a nonempty set. For each A⊆S, let χA be the characteristic functionofAwithvaluesinZ2. i.e.,χA(x)=1forx∈AandχA(x)= 0 for x6∈A. Let F = {χA|A⊆S}, then for B,C ⊆S, the binary operations ⊕S and⊗S are defined on F by (χB⊕SχC)(x) = χB(x)⊕χC(x), (χB⊗SχC)(x) = χB(x)⊗χC(x), in which⊕ and⊗denote the operations of addition and multiplication in Z2 respectively. An algebra of sets on S is a nonempty collectionA⊆P(S) such that, 0/∈A, ∀B,C∈A:B∪C∈A, ∀B∈A:S−B∈A. A finitely additive function

onAis a function m:A→[0,+∞] such that m(B∪C) = m(B)+m(C) for all B,C∈A with B∩C = 0/ [7]. The function m assigns to each element B ofA, an element m(B) of the set of non negative extended real numbers. The finitely additive function m is called bounded if m(S) < +∞. An algebra A⊆P(S) of sets is called a σ−algebra, if every union of any countable collection of sets inAis again inA. A countably additive function monaσ−algebra Aisa functionm:A→[0,+∞] such that m(∪An) =∑m(An) for each sequence An of disjoint sets inA. If m is a countably additive function on an algebra A⊆P(S), then (A,m) is called a measure space. If (A,m) is a measure space and m(S) = 1, then (A,m) is called a probability space. A pseudometric space is a generalized metric space in which the distance between two distinct points can be zero. Apseu dometr icon S is a real function d :S×S→R which satisfies the conditions ∀x,y,z∈S:d(x,y) = d(y,x),d(x,x) = 0,d(x,z)≤d(x,y)+d(y,z). The pseudometric topology is defined exactly as the metric space topology. A pseudometric topology is Hausdorff if and only if it is a metric space [1]. In the following, we are going to establish the aforementioned structures on P(S) 3. Mathematical structure son P(S)

3.1. Making P(S) to a Boolean ring

Let F be the set of all Z2 characteristic functions of the subsets of S. Obviously χS = 1,χ0/ = 0 and χB = χC ⇔B =C. Therefore: Theorem3.1.1The sets P(S) and F are equivalent. Let B4C denotes the symmetric difference of the sets B,C. The proofs of the following two theorems are straightforward. Theorem3.1.2Let B,C⊆S, then χB⊕S χC = χB4C, χB⊗S χC = χB∩C, χS−B = 1⊕S χB.

Theorem3.1.3 Let n∈N and B1,...,Bn ⊆S, then χB1 ⊕S ...⊕S χBn = χB14...4Bn, χB1 ⊗S ...⊗S χBn = χB1∩...∩Bn.

Theorem3.1.4 The triple (F,⊕S,⊗S) is a Boolean unitary ring. Proof. Let B⊆S then χB⊗χB = χB∩B = χB. Theorem 3.1.5 The rings (P(S),4,∩) and (F,⊕S,⊗S) are isomorphic. Proof. The map φ : P(S)→F defined by φ(A) = χA is bijective. Moreover, Theorem 3.1.2 implies that φ(B4C) = χB4C = χB⊕S χC =φ(B)⊕Sφ(C), φ(B∩C) = χB∩C = χB⊗S χC =φ(B)⊗Sφ(C).

3.2. Making P(S) to a measure space Let B∈P(S) and μ(B) is +∞ for an infinite set B and is equal to the number of elements in B for a finite set. Then μ is a countably additive function defined for all subsets of S, therefore: Theorem3.2.1The ordered pair (P(S),μ) is a measure space.

3.3. Making P(S) to a pseudometricspace

Let m be a finitely additive function on the algebra P(S). Theorem 3.2.1 shows that such a function exists. For B,C⊆S let dm(B,C) = m(B4C), then: Theorem3.3.1Theorderedpair (P(S),dm) is a pseudometricspace. Proof. Obviously dm(B,C) = dm(C,B)≥0, B =C⇒dm(B,C) = 0 for all B,C⊆S. For B,C,D⊆S we have dm(B,D)+dm(D,C)−dm(B,C) = m(B4D)+m(D4C)−m(B4C) = m(B−D)+m(D−B)+m(D−C) +m(C−D)−m(B−C)−m(C−B) = [m(B−D)+m(D−C)−m(B−C)] +[m(C−D)+(D−B)−m(C−B)]≥0. Because (C−D)∩(D−B) = (B−D)∩(D−C) = 0/, C−B⊆(C−D)∪(D−B), B−C⊆(B−D)∪(D−C). This completes the proof Let B⊆S and {Bn}n∈N be a sequence of elementsin(P(S),m). Then we write limn→+∞Bn = B if and only if limn→+∞dm(Bn,B) = 0. As a consequence we have: Theorem3.3.2If limn→+∞Bn = B, then limn→+∞m(Bn) = m(B). Proof. Let limn→+∞Bn = B, then limn→+∞d(Bn,B) = 0, so for any ε > 0 there exists n0 ∈ N such that, if n > n0 then m(Bn−B)+m(B−Bn) <ε. But m(Bn) = m(Bn−B)+m(B∩Bn)≤m(Bn−B)+m(B), m(B) =

m(B−Bn)+m(Bn∩B)≤m(B−Bn)+m(Bn).   Therefore|m(Bn)−m(B)|≤m(Bn−B)+m(B−Bn) <ε forn>n0. This completes the proof.

 Theorem 3.3.3 The function m : (P(S),dm) → (R,| |) is continuous. With the preceding notations let B,C⊆S are such that dμ(B,C) =0, then μ(B4C) = 0 . Thus μ(B−C) = μ(C−B) = 0 and B = C. Therefore:

Theorem3.3.4The ordered pair (P(S),dμ) is a metric space.

In the next section, we are going to find the measure of some special kinds of sets in a bounded measure space (P(S),m).

IV. THE MEASURE OF SOME SETS

LetN be the set of natural numbers,O⊆N be these to fodd numbers and E = N−O. For the bounded measure space (P(S),m) and the subsets B1,···,Bn, let On ={a∈S|∃r∈O,r≤n,i1,...,ir ≤n,card{i1,...,ir}= r ,a∈Bi1 ∩...∩Bir}, En ={a∈S|∃r∈E,r≤n,i1,...ir ≤n,card{i1,...,ir}= r ,a∈Bi1 ∩...∩Bir}. Since On and En are disjoint subsets, then m(On)+m(En) = m(S). In the following Theorem, On isinter preddueto some operationsin commutative unitary ring (P(S),4,∩). The proof of the Theorem is a consequence of the associativity property of symmetric difference and a straightforward induction. Theorem4.1Let n∈N and B1,···,Bn are some subsets of S. Then On = B14···4Bn.

The following Theorem is a consequence of Theorem 4.1. Theorem4.2 Let m(Bi) = mi,m(Bi∩Bj) = mij,m(Bi∩Bj∩Bk) = mijk,...,Sn 1 =∑imi,Sn 2 =∑i<jmij,Sn 3 =∑i<j<kmijk ,...forn∈N and B1,...,Bn ∈P(S). Then m(On) =∑n r=1(−2)r−1Sn r. Proof. Let n = 2, then m(O2) = m(B14B2) = m(B1∪B2)−m(B1∩B2) = m(B1)+m(B2)−2m(B1∩B2) = m1 +m2−2m12 = S2 1−2S2 2 = 2 ∑ r=1 (−2)r−1S2 r.

Therefore the Theorem is hold for n = 2. Let n = k and m(Ok) = ∑k r=1(−2)r−1Sk r. If Ci = Bi∩Bk+1 for 1 ≤ i ≤ k, then m(Ci) =m i(k+1),m(Ci∩Cj) = mij(k+1),··· and etc. Thus for n = k+1 we have m(Ok+1) = m(B14···4Bk4Bk+1) = m[(B14···4Bk)4Bk+1] = m(B14···4Bk)+m(Bk+1)−2m[(B14···4Bk)∩Bk+1] = m(B14···4Bk)+m(Bk+1) −2m[(B1∩Bk+1)4···4(Bk∩Bk+1)] = m(Ok)+m(Bk+1)−2m(C14···4Ck)

 This completes the proof.

In the following section, we are supposed to present some applications of Theorem 4.2. 5. Estimationa probability

In this part we estimate the measure of On in a uniform discrete probability space [14]. First of all we prove: Theorem5.1If n∈N, then n! > (n e)n. Proof. If n = 1 then the inequality is hold. Let the Theorem is true for n = k, i.e., k! > (k e)k. A computation using the inequality of the arithmetic and geometric means shows that the sequence{(1+ 1 n)n}n∈N is a bounded increasing sequence with limit e. Therefore (1+ 1 n)n < e for all n∈N. If n = k+1, then the hypothesis of the induction, and the above inequality implies that

This completes the induction.

Theorem 5.2 Let S be the set of all permutations of the set M = {1,···,n},a = 1−e−2 2 , and Bi = {f ∈ S : f(i) = i} for i = 1,···,n. Then m(On) = ∑k=n k=1(−2)k−1(1 k!), and if n≥10 then |m(On)−a|< 2 1000. Moreover, m(On) < 7 15 for n > 4. Proof. A computation using uniform probability laws yields that

so Theorem 4.2 implies that m(On) =∑k=n k=1(−2)k−1(1 k!). Moreover, a computation using Theorem 5.1 shows that for n≥10 we have |m(On)−a|= +∞ ∑ k=n (−2)k (k+1)! < +∞ ∑ k=n (2e k+1)k<+∞ ∑ k=n

2−k = 2−n+1 < 2 1000

.The last inequality in the Theorem, comes from the Leibniz's rule for alternative series [13].

The following examples are some confirmations for the previous Theorem: Example5.3Let n computers with labels 1,...,n are used randomly by n users with labels 1,...,n. Let $1 \leq i \leq n$ and Bi be the event that the i−th computer is used by the user the same label. The probabilities mi,mij,mijk,··· are computed as in the proof of Theorem 5.2. Therefore, the results of the previous theorem, also holds in this case. In particular, if the number of users is bigger than 9, then the probability of the occurrence of an odd number of Bis is less than 0.002. In other words, the probability that an odd (res. even) number of computers are used by the same odd (res. even) number of users with same labels randomly, is less (res. greater) than 0.002 (res. 0.980). Moreover

m(O1) = 1, m(O2) = 0, m(O3) =2,3

Example5.4 An institute of standards and industrial research dispatches randomly two inspectors to two food manufacturer companies once a season. Obviously, the dispatching of exactly one inspector on his previous visited company is impossible. The previous theorem also implies that m(O2) = 1−2(1/2!) = 0.

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