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An inequality for generalized normalized δ-Casorati curvaturesfor bi-slant submanifolds of cosymplectic space form

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ABSTRACT

In this paper, we prove an optimal inequality for the normalized scalar curvature and the generalized normalized δ-Casorati curvatures for bi-slant submanifolds of cosymplectic space forms. Moreover, we characterize those submanifolds for which the equality holds.

Keywords: Casorati curvature, cosymplectic space form, scalar curvature.
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I. INTRODUCTION

The Casorati curvature(extrinsic invariant) of a submanifold of a Riemannian manifold introduced by Casorati defined as the normalized square length of the second fundamental form [2]. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [11]. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [7, 8, 14, 25, 26]. Therefore it attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [17, 18, 23].

In this paper, we will study the inequalities for the generalized normalized δ-Casorati curvature for bi-slant submanifolds of cosymplectic space forms.

II. PRELIMINARIES

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure (ϕ,ξ,η,g) where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η is a one form and g is the Riemannian metric on $\overline{M}$. Then they satisfy

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

These conditions also imply that

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X,\xi),$$

and

$$g(\phi X,Y)+g(X,\phi Y)=0,$$

for all vector fields X,Y in $T\overline{M}$. Where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An almost contact metric manifold $\overline{M}$ is said to be a cosymplectic manifold if

$$(\overline{\nabla}_X\phi)Y=0 \quad \text{and} \quad \overline{\nabla}_X\xi=0. \tag{1}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$. The curvature tensor $\overline{R}$ for cosymplectic space forms is defined as

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X-g(X,Z)Y+\eta(X)\eta(Z)Y\\ &-\eta(Y)\eta(Z)X+\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi\\ &-g(\phi X,Z)\phi Y+g(\phi Y,Z)\phi X+2g(X,\phi Y)\phi Z\}\end{aligned} \tag{2}$$

for all $X,Y,Z\in T\overline{M}$.

Let M be a submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g. We write

$$\phi X=PX+FX$$

for any $X\in TM$, PX and FX denote the tangential and normal parts of ϕX respectively.

The equation of Gauss is given by

$$\overline{R}(X,Y,Z,W)=R(X,Y,Z,W)+g(h(X,Z),h(Y,W))-g(h(X,W),h(Y,Z)) \tag{3}$$

for $X,Y,Z,W\in TM$, where $\overline{R}$ and R represent the curvature tensor of $\overline{M}$ and M respectively.

The squared norm of P at $p\in M$ is defined as

$$\|P\|^2=\sum_{i,j=1}^{n} g^2(\varphi e_i,e_j), \tag{4}$$

where $\{e_1,\ldots,e_n,e_{n+1}=\xi\}$ is any orthonormal basis of the tangent space TM of M.

A submanifold M is said to be a slant submanifold if for any $p\in M$ and a non zero vector $X\in T_pM$, the angle between JX and T_pM is constant, i.e., the angle does not depend on the choice of $p\in M$ and $X\in T_pM$. The angle $\theta\in[0,\frac{\pi}{2}]$ is called the slant angle.

In general, if a submanifold M admits two orthogonal distributions D^{θ_1} and D^{θ_2}, such that
(i) $T\mathcal{N}=D^{\theta_1}\oplus D^{\theta_2}$
(ii) For any $i=1,2$, D^{θ_i} is the slant distribution with slant angle θ_i

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is known as *Bi-slant submanifold*. Naturally, the bi-slant submanifolds is a generalization of semi-slant submanifolds and hemi-slant submanfolds. The invariant, anti-invariant, CR-submanifolds, slant submanifolds appears as particular cases:
(i) If $\theta_1 = \theta_2 = 0$, bi-slant submanifolds is invariant submanifold.

(ii) If $\theta_1 = \theta_2 = \frac{\pi}{2}$, bi-slant submanifolds is anti-invariant submanifold.
(iii) If $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, bi-slant submanifolds is CR-submanifold.
(iv) If $\theta_1 = 0$ and $\theta_2 \neq 0$, bi-slant submanifolds is semi-slant submanifold.
(v) If $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq \frac{\pi}{2}$, bi-slant submanifolds is hemi-slant submanifold.

Suppose M be a bi-slant submanifold of a cosymplectic space form $\overline{M}$. Let $p \in M$ and $\{e_1, \ldots, e_n = \xi\}$ an orthonormal basis of T_pM, with

$$e_1, e_2 = \frac{1}{cos\theta} Pe_1, \ldots, e_{2n_1-1}, e_{2n_1} = \frac{1}{cos\theta} Pe_{2n_1-1}, e_{2n_1+1}, e_{2n_1+2} = \frac{1}{cos\theta} Pe_{2n_1+1},$$
$$\ldots, e_{2n_1+2n_2-1}, e_{2n_1+2n_2} = \frac{1}{cos\theta} Pe_{2n_1+2n_2-1}, e_{2n_1+2n_2+1} = \xi,$$

we have

$$g(\phi e_1, e_2) = g(\phi e_1, \frac{1}{cos\theta} Pe_1) = \frac{1}{cos\theta} g(Pe_1, Pe_1) = cos\theta.$$

Similarly, we have

$$g^2(\phi e_1, e_2) = cos^2\theta,$$

thus, we have

$$g^2(\phi e_i, e_j) = \begin{cases} cos^2\theta_1, & \text{for } i = 1, \ldots, 2n_1 - 1 \\ cos^2\theta_2, & \text{for } i = 2n_1 + 1, \ldots, 2n_1 + 2n_2 - 1 \end{cases} \quad (5)$$

Let M be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ at any $p \in M$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n+1)}.$$

The mean curvature vector denoted by H is defined as

$$H = \frac{1}{n} \sum_{i,j=1}^{n} h(e_i, e_i)$$

and also we put

$$h^{\gamma}_{ij} = g(h(e_i, e_j), e_{\gamma}),\ i, j \in 1, 2, .., n,\ \gamma \in \{n+1, n+2, ..., 2m+1\}.$$

The squared norm of mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\gamma=n+1}^{2m+1} \left(\sum_{i=1}^{n} h^{\gamma}_{ii} \right)^2$$

and the squared norm of second fundamental form h is denoted by $\mathscr{C}$ defined as

$$\mathscr{C} = \frac{1}{n} \sum_{\gamma=n+1}^{2m+1} \sum_{i,j=1}^{n} (h^{\gamma}_{ij})^2$$

known as Casorati curvature of the submanifold.

If we suppose that L is an r-dimensional subspace of TM, $r \geq 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of L. then the scalar curvature of the r-plane section L is given as

$$\tau(L) = \sum_{1 \leq \gamma < \beta \leq r} K(e_{\gamma} \wedge e_{\beta})$$

and the Casorati curvature $\mathscr{C}$ of the subspace L is as follows

$$\mathscr{C}(L) = \frac{1}{r} \sum_{\gamma=n+1}^{2m+1} \sum_{i,j=1}^{n} (h^{\gamma}_{ij})^2.$$

A point $p \in M$ is said to be an invariantly quasi-umbilical point if there exist $2m-n+1$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_{2m+1}$ such that the shape operators with respect to all directions ξ_{γ} have an eigenvalue of multiplicity $n-1$ and that for each ξ_{γ} the distinguished eigen direction is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathscr{C}_p + \frac{n+1}{2n} inf\{\mathscr{C}(L) | L : \text{a hyperplane of } T_pM\} \tag{6}$$

and

$$[\widehat{\delta}_c(n-1)]_p = 2\mathscr{C}_p + \frac{2n-1}{2n} sup\{\mathscr{C}(L) | L : \text{a hyperplane of } T_pM\}. \tag{7}$$

Some authors use the coefficient $\frac{n+1}{2n(n-1)}$ instead of $\frac{2n-1}{2n}$ in the equation(7). It was pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ is not suitable and therefore modified by the coefficient $\frac{2n-1}{2n}$. For a positive real number $t \neq n(n-1)$, the generalized normalized δ-Casorati curvatures $\delta_c(t; n-1)$ and $\widehat{\delta}_c(t; n-1)$ are given as

$$[\delta_c(t;n-1)]_p = t\mathscr{C}_p + \frac{1}{nt}(n-1)(n+t)(n^2-n-t)inf\{\mathscr{C}(L)|L: \text{a hyperplane of } T_pM\},$$

if $0 < t < n^2 - n$, and

$$[\widehat{\delta}_c(t;n-1)]_p = r\mathscr{C}_p + \frac{1}{nt}(n-1)(n+t)(n^2-n-t)sup\{\mathscr{C}(L)|L: \text{a hyperplane of } T_pM\},$$

if $t > n^2 - n$.

III. MAIN THEOREM

Theorem 3.1. *Let M be a $n+1$-dimensional bi-slant submanifold of a cosymplec space form $\overline{M}$ of dimension $2n+1$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t;n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{c}{4}\frac{(n-1)}{n+1} + 3\frac{c}{2}[n_1 cos^2\theta_1 + n_2 cos^2\theta_2]. \tag{8}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t;n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(t;n-1)}{n(n-1)} + \frac{c}{4}\frac{(n-1)}{n+1} + 3\frac{c}{2}[n_1 cos^2\theta_1 + n_2 cos^2\theta_2]. \tag{9}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (8) and (9) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_n\}$ and normal orthonormal frame $\{e_{n+1},\ldots,e_{2m+1}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1,\ldots,m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}a \end{pmatrix}, \qquad S_{n+2} = \cdots = S_m = 0. \tag{10}$$

Proof. Let $\{e_1,\ldots,e_n\}$ and $\{e_{n+1},\ldots,e_{2m+1}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ respectively at any point $p \in M$. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ from (2), we have

$$\begin{aligned}\overline{R}(e_i,e_j,e_j,e_i) &= \frac{c}{4}\{g(e_j,e_j)g(e_i,e_i) - g(e_i,e_j)g(e_j,e_i)\} + \frac{c}{4}\{\eta(e_i)\eta(e_j)g(e_j,e_i) \\ &\quad -\eta(e_j)\eta(e_j)g(e_i,e_i) + \eta(e_j)\eta(e_i)g(e_i,e_j) - \eta(e_i)\eta(e_i)g(e_j,e_j) \\ &\quad -g(\phi e_i,e_j)g(\phi e_j,e_i) + g(\phi e_j,e_j)g(\phi e_i,e_i) + 2g(e_i,\phi e_j)g(\phi e_j,e_i)\}\end{aligned} \tag{11}$$

From Gauss equation and (11), we have

 ijates
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$$\tfrac{c}{4}\{g(e_j,e_j)g(e_i,e_i)-g(e_i,e_j)g(e_j,e_i)\}+\tfrac{c}{4}\{\eta(e_i)\eta(e_j)g(e_j,e_i)$$
$$-\eta(e_j)\eta(e_j)g(e_i,e_i)+\eta(e_j)\eta(e_i)g(e_i,e_j)-\eta(e_i)\eta(e_i)g(e_j,e_j)$$
$$-g(\phi e_i,e_j)g(\phi e_j,e_i)+g(\phi e_j,e_j)g(\phi e_i,e_i)+2g(e_i,\phi e_j)g(\phi e_j,e_i)\}$$
$$=R(e_i,e_j,e_j,e_i)-g(h(e_i,e_i),h(e_j,e_j))+g(h(e_i,e_j),h(e_j,e_i)) \tag{12}$$

By taking summation $1\leq i,j\leq n$, we have

$$2\tau=(n+1)^2\|H\|^2-(n+1)\mathscr{C}+\frac{c}{4}n(n+1)+\frac{c}{4}\{-2n+3\sum_{i,j=1}^{m+1}g^2(\phi e_i,e_j)\} \tag{13}$$

Using (5), we get

$$2\tau=(n+1)^2\|H\|^2-(n+1)\mathscr{C}+\frac{c}{4}n(n-1)) \tag{14}$$
$$+3\frac{c}{4}\sum_{i,j=1}^{m+1}g^2(\phi e_i,e_j)\}.$$

Define the following function, denoted by $\mathscr{Q}$, a quadratic polynomial in the components of the second fundamental form

$$\mathscr{Q}=t\mathscr{C}+a(t)\mathscr{C}(L)-2\tau+\frac{c}{4}\{n(n-1)\}+3\frac{c}{4}\sum_{i,j=1}^{m+1}g^2(\phi e_i,e_j)\}, \tag{15}$$

where L is the hyperplane of T_pM. Without loss of generality, we suppose that L is spanned by $e_1,\dots,e_{n-1}$, it follows from (15) that

$$\mathscr{Q}=\frac{n+t}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\gamma})^2+\frac{a(t)}{n-1}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n-1}(h_{ij}^{\gamma})^2-\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}h_{ii}^{\gamma}\right)^2$$

which can be easily written as

$$\mathscr{Q}=\sum_{\gamma=n+1}^{m}\sum_{i=1}^{n-1}\left[\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)(h_{ii}^{\gamma})^2+\frac{2(n+t)}{n}(h_{in}^{\gamma})^2\right]$$
$$+\sum_{n+1}^{m}\left[2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)\sum_{(i<j)=1}^{n}(h_{ij}^{\gamma})^2-2\sum_{(i<j)=1}^{n}h_{ii}^{\gamma}h_{jj}^{\gamma}+\frac{t}{n}(h_{nn}^{\gamma})^2\right] \tag{16}$$

From(16), we can see that the critical points

$$h^c=(h_{11}^{n+1},h_{12}^{n+1},\dots,h_{nn}^{n+1},\dots,h_{11}^{m},\dots,h_{nn}^{m})$$

of $\mathscr{Q}$ are the solutions of the following system of homogenous equations:

$$\begin{cases} \frac{\partial \mathscr{Q}}{\partial h_{ii}^{\gamma}} = 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)(h_{ii}^{\gamma}) - 2\sum_{k=1}^{n} h_{kk}^{\gamma} = 0 \\ \frac{\partial \mathscr{Q}}{\partial h_{nn}^{\gamma}} = \frac{2t}{n} h_{nn}^{\gamma} - 2\sum_{k=1}^{n-1} h_{kk}^{\gamma} = 0 \\ \frac{\partial \mathscr{Q}}{\partial h_{ij}^{\gamma}} = 4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)(h_{ij}^{\gamma}) = 0 \\ \frac{\partial \mathscr{Q}}{\partial h_{in}^{\gamma}} = 4(\frac{n+t}{n}(h_{in}^{\gamma}) = 0, \end{cases} \quad (17)$$

where $i, j = \{1, 2, \ldots, n-1\}, i \neq j$, and $\gamma \in \{n+1, \ldots, m\}$.

Hence, every solution h^c has $h_{ij}^{\gamma} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathscr{Q}$ is of the following form

$$\mathscr{H}(\mathscr{Q}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right) - 2 & -2 & \ldots & -2 & -2 \\ -2 & 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right) - 2 & \ldots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \ldots & 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right) - 2 & -2 \\ -2 & -2 & \ldots & -2 & \frac{2t}{n} \end{pmatrix},$$

H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\left(4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right), 4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right), \ldots, 4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right)\right),$$

and

$$H_3 = diag\left(\frac{4(n+t)}{n}, \frac{4(n+t)}{n}, \ldots, \frac{4(n+t)}{n}\right).$$

Hence, we find that $\mathscr{H}(\mathscr{Q})$ has the following eigenvalues

$$\lambda_{11} = 0,\ \lambda_{22} = 2\left(\frac{2t}{n} + \frac{a(t)}{n-1}\right),\ \lambda_{33} = \cdots = \lambda_{nn} = 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right),$$

$$\lambda_{ij} = 4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right), \lambda_{in} = \frac{4(n+t)}{n},\ \forall\, i, j \in \{1, 2, \ldots, n-1\},\ i \neq j.$$

Thus, $\mathscr{Q}$ is parabolic and reaches at minimum $\mathscr{Q}(h^c) = 0$ for the solution h^c of the system (17). Hence $\mathscr{Q} \geq 0$ and hence

ijates
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$$2\tau \leq t\mathscr{C} + a(t)\mathscr{C}(L) + \frac{c}{4}\{n(n-1)\} + 3\frac{c}{4}\sum_{i,j=1}^{m+1} g^2(\phi e_i, e_j)\},$$

whereby, we obtain

$$\rho \leq \frac{t}{n(n-1)}\mathscr{C} + \frac{a(t)}{n(n-1)}\mathscr{C}(L) + \frac{c}{4}\frac{(n-1)}{n+1} + 3\frac{c}{2}[n_1 cos^2\theta_1 + n_2 cos^2\theta_2]$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$h_{ij}^{\gamma} = 0,\ \forall\, i,j \in \{1,\ldots,n\},\ i \neq j \text{ and } \gamma \in \{n+1,\ldots,m\} \tag{18}$$

and

$$h_{nn}^{\gamma} = \frac{n(n-1)}{t}h_{11}^{\gamma} = \cdots = \frac{n(n-1)}{t}h_{n-1n-1}^{\gamma}, \forall \gamma \in \{n+1,\ldots,m\}. \tag{19}$$

From (18) and (19), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\overline{M}$, such that the shape operator takes the form (10) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

□

Corollary 3.2. *Let M be an $n+1$-dimensional bi-slant submanifold of a cosymplectic space form . Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + \frac{c}{4}\frac{(n-1)}{n+1} + 3\frac{c}{2}[n_1 cos^2\theta_1 + n_2 cos^2\theta_2].$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_{n+1}\}$ and normal orthonormal frame $\{e_{n+2},\ldots,e_{2m+1}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1,\ldots,m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2a \end{pmatrix},\ S_{n+2} = \cdots = S_m = 0. \tag{20}$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \frac{c}{4}\frac{(n-1)}{n+1} + 3\frac{c}{2}[n_1 cos^2\theta_1 + n_2 cos^2\theta_2].$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_{n+1}\}$ a and normal orthonormal frame $\{e_{n+2},\ldots,e_{2m+1}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1,\ldots,m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2a & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, S_{n+2} = \cdots = S_m = 0. \tag{21}$$

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