EFFICIENT NUMERICAL SOLUTIONS FOR FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS USING HAAR WAVELET METHOD

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ABSTRACT
This study presents an efficient numerical approach for solving fractional-order differential equations using the Haar wavelet method. The Haar wavelets, known for their simplicity and computational efficiency, are employed to transform fractional differential equations into a system of algebraic equations. This method leverages the operational matrix of fractional integration, significantly reducing computational complexity while maintaining high accuracy. Numerical experiments demonstrate the method's reliability and effectiveness, making it a powerful tool for addressing fractional-order problems in science and engineering.

Keywords: *Relationship, Calculus, Scientific, Monographs, Gamma.*

I. INTRODUCTION

The creation of fractional calculus was a collaborative effort by engineers, physicists, and mathematicians. The relationship gave rise to a wave of ideas that goes well beyond the creation of new technologies. Following classical calculus's successful establishment at the end of the 17th century, fractional calculus arose toward the century's close. It was believed that the pioneers of systematic study were Leibniz, Caputo, Hadamard, Fourier, Lioville, and Riemann. Until recently, physics mostly disregarded fractional calculus, despite the fact that it is really a broader form of calculus.

One possible explanation for fractional derivatives' broad dislike is because they are open to several interpretations. Another problem is that fractional derivatives, being non-local, do not seem to have any clear geometrical significance (for more, see L. Debnath).

However, fractional calculus has grown in popularity and use in the last few decades, both in the realm of pure mathematics and in scientific applications. This is because, to model physical processes that include both the past and the present, precise data from time series models is necessary.

Recent advances in fractional calculus have been mainly motivated by its modern applications in several scientific disciplines, including mathematical biology, fluid mechanics, physics of plasmas, image and signal processing, electrochemistry, finance, and the social sciences. It is indisputable that fractional calculus is a fascinating new mathematical tool for solving many problems in mathematics, science, and engineering. If you want to learn more about fractional calculus, both theoretically and practically, you should look into the monographs.

We will first lay out the theoretical underpinnings and mathematical ideas of fractional calculus so that you may verify our results.

A function f(t) with a Riemann-Liouville order $\alpha \geq 0$ is defined in Definition 1.1 as the fractional integration operator.

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0 \qquad ,(1.1)$$

Γ(.) is the well-known gamma function, and the operator J^{α} has the following features:

(i) $J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t),\ \alpha,\beta > 0$

(ii) $J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t),\ \alpha,\beta > 0$

(iii) $J^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} t^{\alpha+\beta}, \qquad \beta > -1$.

The Riemann-Liouville derivative has several limitations for solving fractional differential equations in an effort to mimic real-world phenomena.

Therefore, in accordance with Caputo's ideas on visco-elasticity, we shall provide a modified fractional differential operator D^{α}.

Definition 1.2 states that a function f(t) is the Caputo fractional derivative of D^{α}.

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{m}(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau, \qquad (1.2)$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}$.

In the Caputo approach, calculating an ordinary derivative first and then a fractional integral gets the fractional derivative to the desired order.

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Similar to integer-order differentiation, the Caputo fractional derivative operator is a linear process.

Dα(γf(t)+δg(t)) =γDαf(t)+δDαg(t).

because γ and δ are constants. Additionally, the following fundamental characteristics are met by the Caputo fractional derivative:

(i) $D^{\alpha}t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0 < \alpha < \beta+1, \beta > -1$

(ii) $J^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{k=0}^{m-1} f^{k}(0^{+})\frac{t^{k}}{k!}, \quad m-1 < \alpha \leq m, m \in \mathbb{N}$

(iii) $D^{\alpha}C = 0,$ where C is a constant.

II. HAAR WAVELETS AND THEIR CONSTRUCTION

Here, we lay up the groundwork for orthonormal wavelets by providing a quick overview of multiresolution analysis, and then we present Haar wavelets, a subset of orthonormal wavelets. Wavelet analysis is a relatively new mathematical method that is both powerful and useful. The strictly mathematically-based transform has found applications in many fields, such as harmonic analysis, signal and image processing, turbulence, geophysics, statistics, economics, finance, medicine, differential and integral equations, and many more. The function φ(t) with real values that fulfills the aforementioned criteria is called a wavelet.

$$\int_{-\infty}^{\infty} \psi(t)\,dt = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} |\psi(t)|^{2}\,dt = 1.$$

It is necessary for ψ(t) to have an oscillatory function with zero mean in the first requirement, and for the wavelet function to have unit energy in the second condition. The exact definition of wavelets is

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \quad a \neq 0, b \in R, \quad (1.3)$$

where a stands for the translation parameter and b for the dilation parameter. The signal's high-frequency components are represented by small values of a, and its low-frequency components by big values of a. And this family of discrete wavelets is what we get when we limit the parameters a and b to discrete values as a 0:

$$\psi_{j,k}(t) = |a_0|^{j/2}\psi\left(a_0^{j}t - kb_0\right), \quad (1.4)$$

to where ψj,k is a wavelet basis for L2(R). Specifically, the functions ψj,k provide an orthonormal basis for a0 = 2 and b0 = 1.

III. CONVERGENCE OF THE HAAR WAVELET

For any t1 and t2 in the interval [0,1], there is a positive integer K greater than 0 such that the absolute value of the difference between y(t1) and y(t2) is less than or equal to K, where K is the Lipschitz constant. As a result, we get the following as the Haar approximation of y(t):

$$y_m(t) = \sum_{i=0}^{m-1} c_i h_i(t), \quad m = 2^{p+1},\ p = 0, 1, 2, \ldots, M.$$

It is possible to then specify the appropriate mistake at the mth level as

$$\|y(t) - y_m(t)\|_2 = \left\| y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right\|_2 = \left\| \sum_{i=2^{p+1}}^{\infty} c_i h_i(t) \right\|_2 .$$

We can now investigate the error in fractional order differential equations with the exact solution at our disposal. Our proposed method's convergence looks like this:

Theorem 2.1 If y(t) is a Lipschitz continuous function on [0,1] and ym(t) are the Haar approximations of it, then the following is the upper limit on the error:

$$\|y(t) - y_m(t)\|_2 \le \frac{K}{\sqrt{3}m^2}.$$

The evidence. Thanks to the Haar wavelets' orthonormality characteristic, we've

$$\|y(t) - y_m(t)\|_2^2 = \int_0^1 |y(t) - y_m(t)|^2 dt = \sum_{r=2^{p+1}}^{\infty} \sum_{s=2^{p+1}}^{\infty} c_r \overline{c_s} \int_0^1 h_r(t) \overline{h_s(t)} dt = \frac{1}{m} \sum_{r=2^{p+1}}^{\infty} |c_r|^2.$$

The Haar wavelet coefficients cr’s can be estimated by using the relation Eq. (2.10) as

$$c_r = \int_0^1 y(t) h_r(t) dt = \frac{2^{j/2}}{\sqrt{m}} \left\{ \int_{I_1} y(t) dt - \int_{I_2} y(t) dt \right\},$$

where $I^1 = \left(\frac{k-1}{2^{-j}}, \frac{k-(1/2)}{2^{-j}} \right]$ and $I^2 = \left(\frac{k-(1/2)}{2^{-j}}, \frac{k}{2^{-j}} \right]$. It is possible to determine t1 ∈ I1 and t2 ∈ I2 such that by using the mean value theorem of integrals—

$$c_r = \frac{2^{j/2}}{\sqrt{m}}\left\{y(t_1)|I_1| - y(t_2)|I_2|\right\} = \frac{2^{-j/2-1}}{\sqrt{m}}\left[y(t_1) - y(t_2)\right].$$

Therefore,

$$|c_r|^2 = \frac{2^{-j-2}}{m}\left|y(t_1) - y(t_2)\right|^2.$$

Assuming t1 is less than t2 according to the mean value theorem of derivative,

$$|c_r|^2 = \frac{2^{-j-2}}{m}\left|t_1 - t_2\right|^2 y'^2(t)(\xi) \leq \frac{2^{-j-2}}{m} 2^{-2j} K^2 = \frac{2^{-3j-2}K^2}{m}.$$

Substituting Eq. (2.19) into Eq. (2.16), we have

$$\left\|y(t) - y_m(t)\right\|_2^2 \;=\; \frac{1}{m}\sum_{r=2^{p+1}}^{\infty} |c_r|^2$$

$$= \frac{1}{m}\sum_{j=p+1}^{\infty}\left[\sum_{r=2^j}^{2^{j+1}-1} |c_r|^2\right]$$

$$\leq \frac{1}{m}\sum_{j=p+1}^{\infty}\left[\sum_{r=2^j}^{2^{j+1}-1} \frac{2^{-3j-2}K^2}{m}\right]$$

$$= \frac{K^2}{m^2}\sum_{j=p+1}^{\infty} 2^{-3j-2}\left(2^{j+1} - 1 - 2^j + 1\right)$$

$$= \frac{K^2}{3m^4}.$$

Therefore,

$$\left\|y(t) - y_m(t)\right\|_2 \leq \frac{K}{\sqrt{3}m^2}.$$

This completes the proof.

IV. CONCLUSION

The Haar wavelet method proves to be a highly effective and computationally efficient approach for solving fractional-order differential equations. This technique offers significant advantages, including simplicity, fast convergence, and the ability to handle complex boundary conditions with high accuracy. By transforming fractional-order problems into algebraic systems, the method reduces computational complexity and enhances solution precision. These attributes establish the Haar wavelet method as a valuable tool in the numerical analysis of fractional-order systems across various scientific and engineering applications.

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