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# Analysis of Thermal Stresses in a Transversely Isotropic Annular Cylinder Heated by γ Radiation

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**Abstract**

*The aim of this paper is to analyze thermal stresses in an inhomogeneous transversely isotropic long hollow cylinder, where the outer curved surface is perfectly insulated, and heat is generated due to γ-ray irradiation. The material inhomogeneity is modeled by assuming that the elastic modulus and the coefficient of thermal expansion vary as the nth power of the radial distance, while the coefficient of thermal conductivity varies linearly with the radial distance. The variation of hoop stress at the inner wall of the cylinder is presented graphically for different values of cylinder thickness.*

## 1. Introduction

The study of non-homogeneity in aeolotropic (anisotropic) materials has recently become a significant focus in the mechanics of solids. Due to manufacturing processes and various technological factors, elastic solid bodies often exhibit not only anisotropy but also different forms of material non-homogeneity.

Considerable attention has been given by researchers to the analysis of thermal stresses in isotropic cylinders subjected to internal heat generation caused by axisymmetric radiation. Well [15] addressed the problem of determining thermal stresses in isotropic cylinders with perfectly insulated outer curved surfaces, where the source of heat generation is γ-ray irradiation. Bagchi [1] extended this study to non-isotropic materials. Mollah [7] further investigated the problem for an in-homogeneous transversely isotropic long hollow cylinder, assuming that the elastic coefficients, coefficient of thermal expansion, and thermal conductivity vary linearly with the radial distance.

The present paper generalizes the aforementioned work. Here, the material non-homogeneity is modelled by assuming that both the elastic coefficients and the coefficient of thermal expansion vary as the $n$th power of the radial distance $r$, while the thermal conductivity varies linearly with $r$.

Furthermore, the material of the long hollow cylinder is considered to be a transversely isotropic elastic solid, with its outer curved surface perfectly insulated. The internal heat is assumed to be generated due to γ-ray radiation. Numerical results are also presented for magnesium, showing that the hoop stress at the inner boundary increases progressively with the thickness of the cylinder for arbitrary values of the absorption constant.

## 2. Formulation, temperature profile and assumptions

We use a cylindrical coordinate system, with the *z*-axis aligned along the axis of the cylinder. Let the temperature distribution be axisymmetric and independent of the axial coordinate. The rate *H* at which heat is generated within the cylinder varies according to the law given in [4].

$$H = H_i e^{-\mu(r-a)} \tag{1}$$

where

$H_i$ = heat generation rate on the inside wall of the cylinder,

$a$ = inside radius and

$\mu$ = the absorption coefficient for the γ-ray energy.

For the present problem the temperature $T$ satisfies the conductivity equation Love [5]

$$K\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) + \frac{dK}{dr}\frac{dT}{dr} = H_1 e^{-\mu(r-a)} \tag{2}$$

where K is the terminal conductivity of the material obeying the law

$$K = K_0 r \tag{3}$$

where $K_0$ being a non-zero positive constant.

Using (3) we get from (2)

$$r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = -\frac{H_1}{K_0} e^{-\mu(r-a)} \tag{4}$$

Since the outer wall is insulated and the inner wall is maintained at a constant temperature, the boundary conditions are as follows:

$$\left.\begin{array}{l} T = T_1 \text{ (constant)} \quad on \quad r = a \\ \text{and} \\ \dfrac{dT}{dr} = 0 \quad on \quad r = b \end{array}\right\} \tag{5}$$

The solution of the equation (4) subject to the condition (5) is

$$T = T_1 + p\left[\frac{1-\lambda}{a}\right] + \frac{\lambda}{r} - \frac{1}{r}e^{-\mu(r-a)} \tag{6}$$

where

$$\lambda = (1 + \mu b)\, e^{-\mu(b-a)} \text{ and } p = \frac{H_1}{K_0 \mu^2} \tag{7}$$

### 3. Mechanical stress patterns:

Assuming the axially symmetric character of the problem, the non-vanishing components of stress tensors are $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$, and $\sigma_{rz}$. Thus the stress-strain relations for transversely isotropic material are given by [13]

$$\left.\begin{aligned}
\sigma_{rr} &= c_{11}' e_{rr} + c_{12}' e_{\theta\theta} + c_{13}' e_{zz} - b_1' T \\
\sigma_{\theta\theta} &= c_{12}' e_{rr} + c_{11}' e_{\theta\theta} + c_{13}' e_{zz} - b_1' T \\
\sigma_{zz} &= c_{13}' e_{rr} + c_{13}' e_{\theta\theta} + c_{33}' e_{zz} - b_2' T \\
\sigma_{rz} &= c_{44}' e_{rz}
\end{aligned}\right\} \tag{8}$$

Where

$$\left.\begin{aligned}
b_1' &= \left(c_{11} + c_{12}'\right)\alpha_1' + c_{13}'\alpha_2' \\
b_2' &= 2c_{13}'\alpha_1' + c_{33}'\alpha_2'
\end{aligned}\right\} \tag{9}$$

are elastic coefficients and for inhomogeneity they are assumed to be functions of $r$, $T$ is the temperature at a point ($r, \theta, z$) and $\alpha_1'$ and $\alpha_2'$ are the coefficients of linear thermal expansion along and perpendicular to the z-axis respectively.

Now, the strain components for the problem are given by

$$e_{rr} = \frac{du}{dr}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = 0 \quad and \quad e_{rz} = 0, \tag{10}$$

where $u_r = u, \ u_\theta = 0, \ u_z = 0.$

For non-homogeneity of the material we assume

$$c_{ij}' = c_{ij} r^n, \quad and \quad \alpha_i' = \alpha_i r^n, \tag{11}$$

where $c_{ij}$ and $\alpha_i$ are non-zero positive constants.

The relations (8) with (10) and (11) give

$$\left.\begin{aligned}
\sigma_{rr} &= c_{11} r^n \frac{du}{dr} + c_{12} r^{n-1} u - b_1 r^{2n} T, \\
\sigma_{\theta\theta} &= c_{12} r^n \frac{du}{dr} + c_{11} r^{n-1} u - b_1 r^{2n} T, \\
\sigma_{zz} &= c_{13} r^n \frac{du}{dr} + c_{13} r^{n-1} u - b_2 r^{2n} T,
\end{aligned}\right\} \tag{12}$$

where

$$\left.\begin{aligned}
b_1 &= \left(c_{11} + c_{12}\right)\alpha_1 + c_{13}\alpha_2, \\
b_2 &= 2c_{13}\alpha_1 + c_{33}\alpha_3.
\end{aligned}\right\} \tag{13}$$

The stress equations of equilibrium in the absence of the body forces are [14]

$$\left.\frac{\partial \sigma_{rr}}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \quad \text{and} \quad \frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=0\right\} \qquad \textbf{14)}$$

The second equation of (14) automatically holds and the first, by (12) and (6) becomes to

$$r^2\frac{d^2u}{dr^2}+(n+1)\,r\frac{du}{dr}+\left(n\frac{c_{12}}{c_{11}}-1\right)u$$
$$=\frac{b_1}{c_{11}}\left[2\left\{n\left(T_1+\frac{1-\lambda}{a}p\right)\right\}r^{n+1}+(2n-1)\lambda p r^n\right.$$
$$\left.-p\{(2n-1)-\mu r\}r^n e^{-\mu(r-a)}\right] \qquad \textbf{(15)}$$

The particular integral of the equation (15) is [16]

$$u=\frac{b_1}{c_{11}}\left[\frac{2n\left(T_1+\frac{1-\lambda}{a}p\right)}{(n+1)(2n+1)+s}r^{n+1}+\frac{(2n-1)\lambda p}{2n^2+s}r^n\right.$$
$$+\frac{pe^{\mu a}}{(\beta_1-\beta_2)}\left\{r^{\beta_1}\times\int_a^r r^{-\beta_1+(n-1)}\{(2n-1)-\mu r\}e^{-\mu r}dr\right.$$
$$\left.\left.+r^{\beta_2}\times\int_a^r r^{-\beta_2+(n-1)}\{(2n-1)-\mu r\}e^{-\mu r}dr\right\}\right]$$

where

$$\beta_1,\beta_2=\frac{-n\pm\sqrt{n^2-4s}}{2}, \qquad s=n\frac{c_{12}}{c_{11}}-1$$

Thus, the general solution of (15) is given by

$$u=A_1r^{\beta_1}+A_2r^{\beta_2}+\frac{b_1}{c_{11}}\left[\frac{2n\left(T_1+\frac{1-\lambda}{a}p\right)}{(n+1)(2n+1)+s}r^{n+1}+\frac{(2n-1)\lambda p}{2n^2+s}r^n\right.$$
$$+\frac{pe^{\mu a}}{(\beta_1-\beta_2)}\left\{r^{\beta_1}\int_a^r r^{-\beta_1+(n-1)}((2n-1)-\mu r)e^{-\mu r}dr\right.$$
$$\left.\left.+r^{\beta_2}\int_a^r r^{-\beta_2+(n-1)}((2n-1)-\mu r)e^{-\mu r}dr\right\}\right] \qquad \textbf{(16)}$$

where $A_1$ and $A_2$ are constants.

Hence the stresses as calculated from (12) are

$$\left.\begin{aligned}
\sigma_{rr} &= c_{11}\left[m_1 A_1 r^{\beta_1+(n-1)} + m_2 A_2 r^{\beta_2+(n-1)}\right] \\
&+ b_1\left[\frac{s}{s+6}\left(T_1+\frac{1-\lambda}{a}p\right)r^{2n} + \frac{pe^{\mu a}}{(\beta_1-\beta_2)}\left\{m_1F_1(r)-m_2F_2(r)\right\}\right] \\
&+\left\{\frac{(2n\text{-}1)}{2n^2+s}-b_1\right\}\lambda p r^{2n-1} + b_1 p e^{-\mu(r-a)}\, r^{2n-1} \\
\sigma_{\theta\theta} &= c_{12}\left[m_1 A_1 r^{\beta_1+(n-1)} + m_2 A_2 r^{\beta_2+(n-1)}\right] + b_1\left[k_1\left(T_1+\frac{1-\lambda}{a}p\right)r^{2n}\right. \\
&\left. + \frac{pe^{\mu a}}{(\beta_1-\beta_2)}\left\{(\beta_1+1)F_1(r)-(\beta_1+1)F_2(r)\right\}\right] + b_1 p e^{-\mu(r-a)}\, r^{2n-1}
\end{aligned}\right] \quad \textbf{(17)}$$

where $m_i = s+1+\beta_i, \quad i = 1, 2$

$$n_i = \beta_i + \frac{n}{s+1}$$

$$k_1 = \frac{c_{12}}{c_{11}}\left[\frac{2n\,(n+1)}{(n+1)(2n+1)+s} + \frac{2\quad n}{(n+1)(2n+1)+s}\right]$$

$$F_i(r) = r^{\beta_i+(n-1)}\int_a^r r^{-\beta_i+(n-1)}\left\{(2n-1)-\mu r\right\}e^{-\mu r}\,dr$$

so that $F_i(a) = 0, \qquad (i = 1, 2)$

A distribution of normal force, as given by equation (17), is required to be applied at the ends of the cylinder in order to maintain w=0 throughout the structure.

Let us assume that an axial stress $\sigma_{zz} = c_1$ (constant) (a constant) acts on the system. By choosing $c_1$ appropriately, we can ensure that the **resultant axial force** on the ends becomes zero. According to **Saint-Venant's principle**, this specific distribution of axial stress will produce **localized effects** only near the ends of the cylinder.

Due to the **superposition** of the uniform axial stress $c_1$ the radial and hoop stresses, $\sigma_{rr}$, and $\sigma_{\theta\theta}$ remain unaffected. However, the **axial displacement** u is influenced. This is the **generalized form of the result obtained by Mollah [7]** in this context.

Accordingly, a term $c_1/c_{13}$ must be added to the expression for u in equation (16).

Setting aside the detailed analysis of displacement, we proceed to apply the **boundary conditions** in order to determine the constants $A_1$ and $A_2$ for our specific problem.

$$\sigma_{rr} = 0, \quad \text{on } r = a \text{ and } r = b \ (a < b) \qquad \textbf{(18)}$$

Using the boundary conditions (18) and the expression for $\sigma_{rr}$ as given in (17) we get

$$\left.\begin{aligned} A_1 &= b_1 P\, \frac{L_2 a^{\beta_2 + (n-1)} - L_1 b^{\beta_2 + (n-1)}}{c_{11} m_1} \\ A_2 &= b_1 P\, \frac{L_1 b^{\beta_1 + (n-1)} - L_2\, a^{\beta_1 + (n-1)}}{c_{11} m_1} \end{aligned}\right\} \qquad \textbf{(19)}$$

with

$$P = \frac{1}{a^{\beta_1 + (n-1)}\, b^{\beta_2 + (n-1)} - a^{\beta_2 + (n-1)}\, b^{\beta_1 + (n-1)}}$$

where

$$L_1 = \frac{s-(n-1)}{(n+1)\,(2n+1)+s} a^{2n}\, T_1 + \left\{(1-\lambda)\frac{s-(n-1)}{(n+1)\,(2n+1)+s} + \lambda \frac{(n-1)\{s-(n-1)\}}{n\left(2n^2+s\right)} + 1\right\} a^{2n-1} p$$

$$L_2 = \frac{s-(n-1)}{(n+1)\,(2n+1)+s} b^{2n} T_1 + \left\{(1-\lambda)\frac{\{s-(n-1)\}b^{2n}}{a\,(n+1)\,(2n+1)+s} + \lambda \frac{(n-1)\{s-(n-1)\}}{n\left(2n^2+s\right)} b^{2n-1} + b^{2n-1} e^{-\mu(b-a)} + f_1(b)\right\} p$$

and $$f_1(r) = \frac{e^{\mu a}}{\left(\beta_1 - \beta_2\right)}\left[m_1 F_1(r) - m_2 F_2(r)\right]$$

$$f_1(b) = \frac{e^{\mu a}}{\left(\beta_1 - \beta_2\right)}\left[m_1 F_1(b) - m_2 F_2(b)\right]$$

Substituting the values of $A_1$ and $A_2$ we get the stress components from (17) as follows:

$$\frac{\sigma_{rr}}{b_1} = \frac{s-(n-1)}{(n+1)\ (2n+1)+s} T_1 \varphi_1(r) + p\varphi_2(r) \tag{20}$$

where $p = \dfrac{H_1}{K_0 \mu^2}$

$$\varphi_1(r) = P\left(R_1 a^{2n} + R_2 b^{2n}\right) + r^{2n}$$

$$\varphi_2(\mathrm{r}) = P\left[R_1\left\{(1-\lambda)\frac{5(n-1)}{(n+1)\ (2n+1)+s} + \lambda\frac{(n-1)\{s-(n-1)\}b^{2n}}{n\left(2n^2+s\right)} + 1\right\}a^{2n-1}\right.$$

$$\left. + R_2 S' + \frac{(1-\lambda)\{s-(n-1)\}b^{2n}}{a(n+1)\ (2n+1)+s} r^{2n}\right]$$

$$+ \lambda\frac{(n-1)\{s-(n-1)\}}{n\left(2n^2+s\right)} r^{2n-1} + e^{-\mu(r-a)} r^{2n-1} + f_1(r)$$

$$S' = \frac{1-\lambda}{a}\frac{\{s-(n-1)\}}{(n+1)(2n+1)+s} + \lambda\frac{(n+1)\{s-(n-1)\}}{n\left(2n^2+s\right)}$$

$$+ b^{2n-1} e^{-\mu(b-a)} + f_1(b)$$

$$R_1 = b^{\beta_1 + (n-1)} r^{\beta_2 + (n-1)} - b^{\beta_2 + (n-1)} r^{\beta_1 + (n-1)}$$

$$R_2 = a^{\beta_2 + (n-1)} r^{\beta_1 + (n-1)} - a^{\beta_1 + (n-1)} r^{\beta_2 + (n-1)}$$

$$\frac{\sigma_{\theta\theta}}{b_1} = \frac{c_{12}}{b_1}\left[n_1\psi_1(r) + n_2\psi_2(r)\right] + k_1\left(T_1 + \frac{1-\lambda}{a}p\right)r^{2n}$$
$$+ \frac{e^{\mu a}}{\beta_1 - \beta_2}\left[\left\{\left(\beta_1 + 1\right)F_1(r) + \left(\beta_2 + 1\right)F_2(r)\right\}\right.$$
$$\left. + \frac{n(2n-1)}{2n^2 - S} - b_1\right]\lambda p r^{2n-1} + p e^{-\mu(r-a)} r^{2n-1} \qquad \textbf{(21)}$$

where

$$\psi_1(r) = r^{\beta_1 + (n-1)} + b_1 P \frac{L_2 a^{\beta_2 + (n-1)} - L_1 b^{\beta_2 + (n-1)}}{c_{11} m_1}$$

$$\psi_2(r) = r^{\beta_2 + (n-1)} + b_1 P \frac{L_1 b^{\beta_1 + (n-1)} - L_2 a^{\beta_1 + (n-1)}}{c_{11} m_2}$$

**4. Computational Results and Analysis:**

We now consider some numerical results for the following range of parameters**:** 10 per cm. $\leq \mu \leq$ 30 per cm., 1.5 cm. < b < 6.0 cm. and a =1 cm.

We consider the material to be made up of magnesium for which the elastic constants on the inner boundary $r = a = 1$ are given by [17]

$$c_{11}' = c_{11} = 0.565 \times 10^{12}\, Dynes / cm^2$$

$$c_{12}' = c_{12} = 0.232 \times 10^{12}\, Dynes / cm^2$$

$$c_{13}' = c_{13} = 0.181 \times 10^{12}\, Dynes / cm^2$$

$$c_{33}' = c_{33} = 0.587 \times 10^{12}\, Dynes / cm^2$$

$$c_{44}' = c_{44} = 0.168 \times 10^{12}\, Dynes / cm^2$$

The coefficients of linear thermal expansion of the said material on the inner boundary $r = a = 1$ are

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$$\alpha_1' = \alpha_1 = 27.7 \times 10^{-6} \text{cms}/^\circ \text{C}$$
$$\alpha_2' = \alpha_2 = 26.6 \times 10^{-6} \text{cms}/^\circ \text{C}$$

Further we choose arbitrarily $T_1 = 500^\circ C$ and $H_1 = 1$.

Using the above data we calculate

$$\alpha \left[ = 10^{-11} \sigma_{\theta\theta}|_{r=a=1} \right]$$

and show graphically for different values of $b$ and $\mu = 20$ in Fig.-2.6. It is seen that the hoop-stress increases as the radius of the outer circular section of the annulus increases.

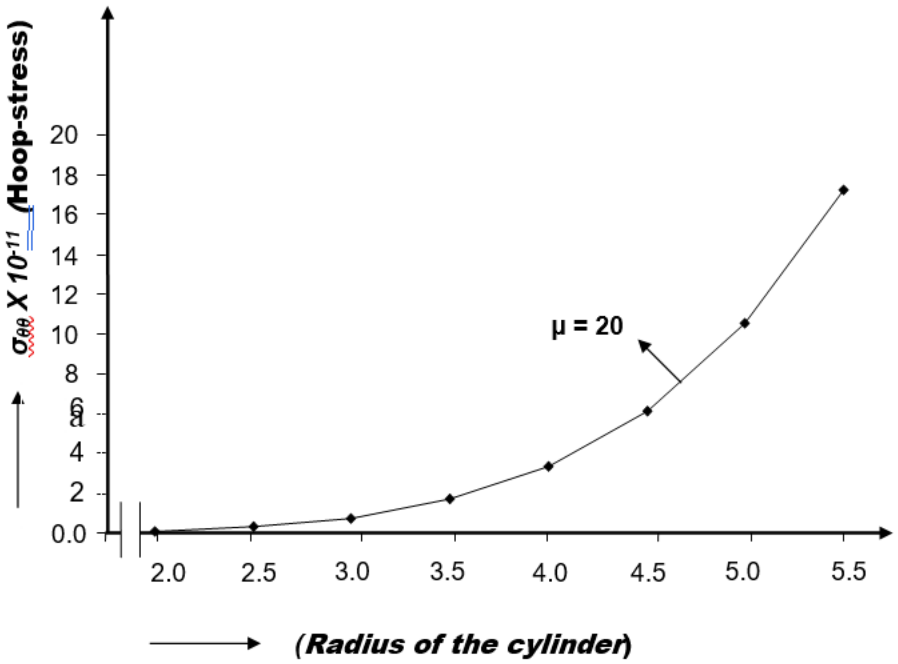


***Fig.-2.6: Variation of the hoop-stress on the inner wall with the radial distance of the cylinder when*** **μ = 20.**

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